A mathematical excursion on the topic of....

Pick's Theorem & Ehrhart Polynomials

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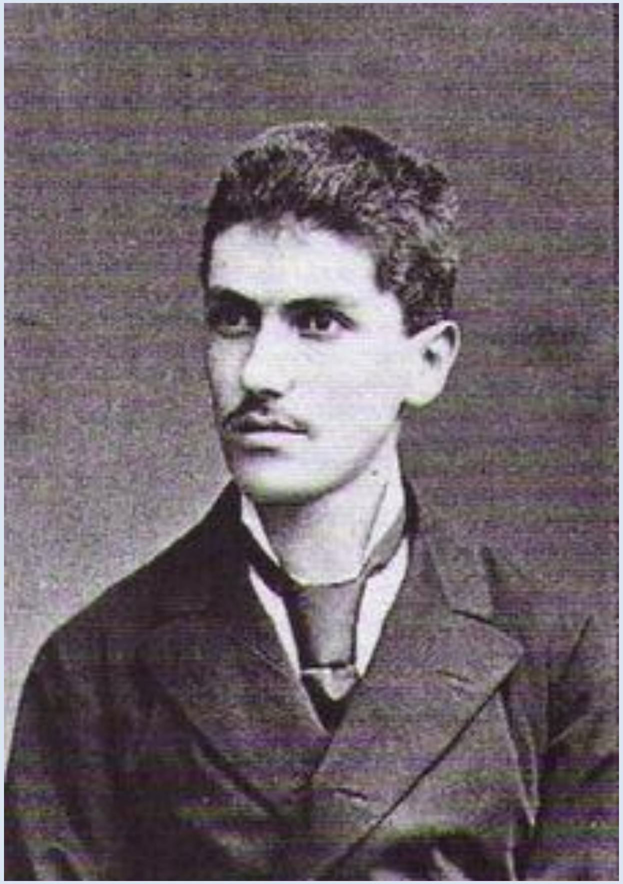
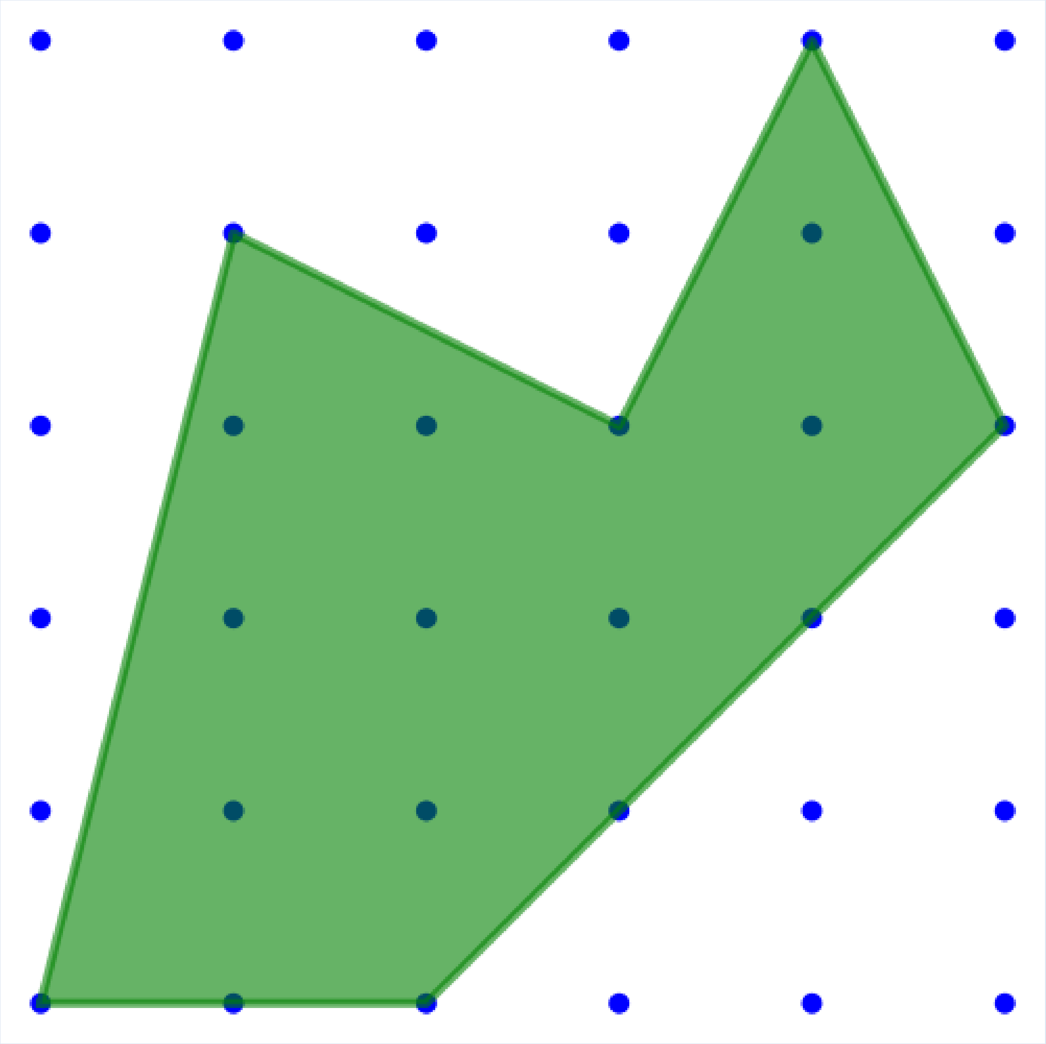
Pick’s Theorem



In 1899, Georg Pick found a single, simple formula for calculating the area of many different shapes.

Pick’s Theorem



A *lattice polygon* is a 2d shape (without holes) built from straight lines, whose corners all have integer coordinates.

Pick's Theorem

Pick's Theorem

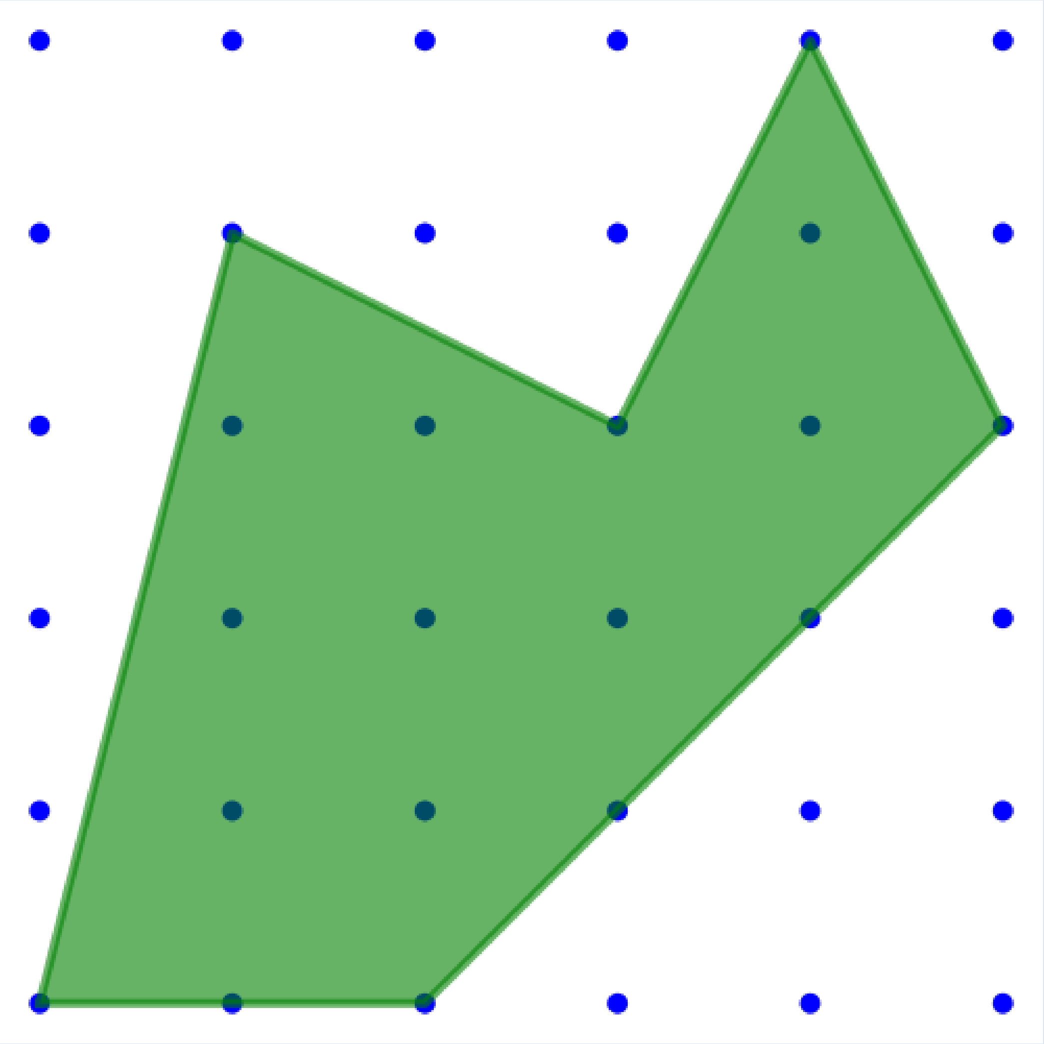
Let P be a lattice polygon.

Suppose that P contains C lattice points in its interior, and B on its boundary.

Then the area (A) of P is:

$$A = C + \tfrac{1}{2}B - 1$$

Pick's Theorem



Here, $B = C = 9$.

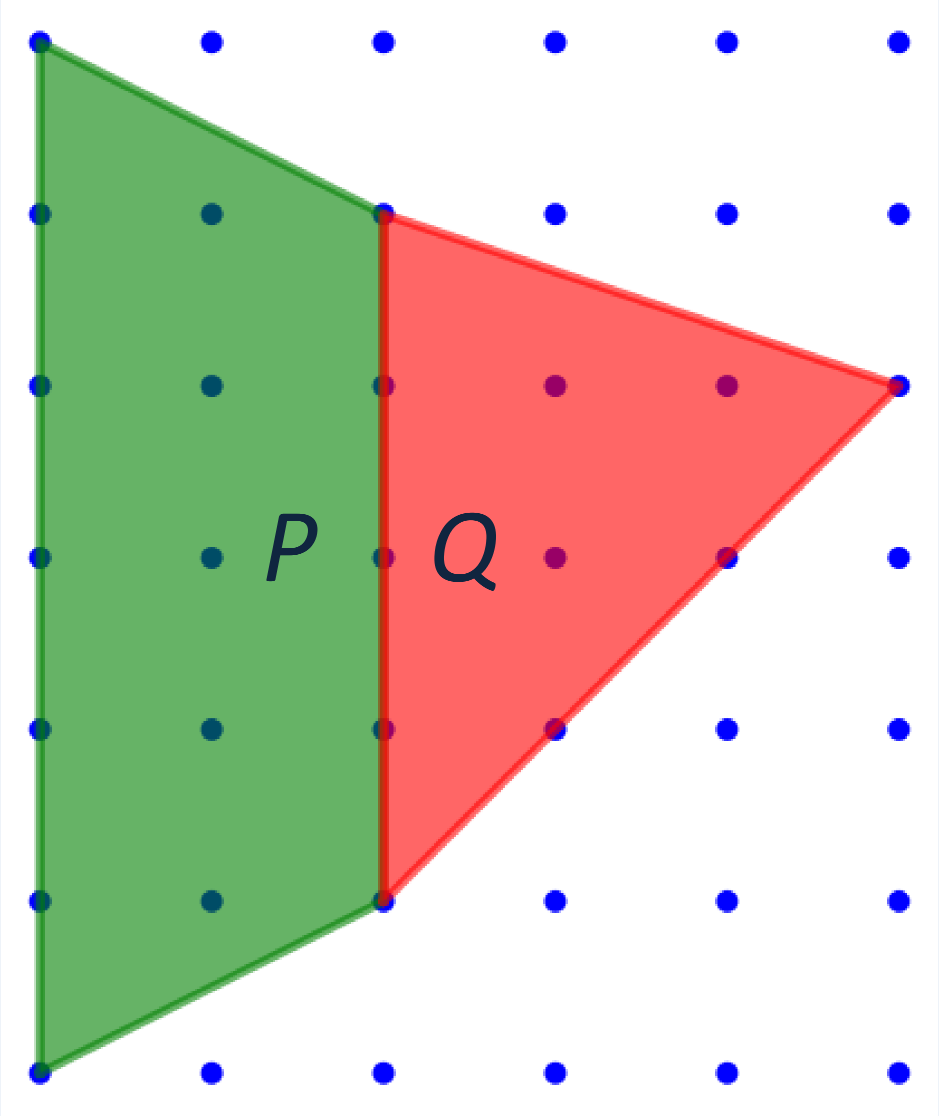
So,

$$A = 9 + (\tfrac{1}{2} \times 9) - 1$$
$$= 12\tfrac{1}{2}.$$

Pick's Theorem

Sketch Proof

Step 1: Show that if *P* & *Q* each satisfy the theorem then so does *R*, which consists of *P* & *Q* joined along an edge.



Pick's Theorem

Proof of Step 1. We are supposing that

$$A_P = C_P + \tfrac{1}{2}B_P - 1$$

&

$$A_Q = C_Q + \tfrac{1}{2}B_Q - 1$$

Clearly also, $A_R = A_P + A_Q$.

So

$$A_R = (C_P + C_Q) + \tfrac{1}{2}\,(B_P + B_Q) - 2 \qquad [i]$$

Pick's Theorem

Suppose there are *D* points on the edge joining *P* & *Q*.

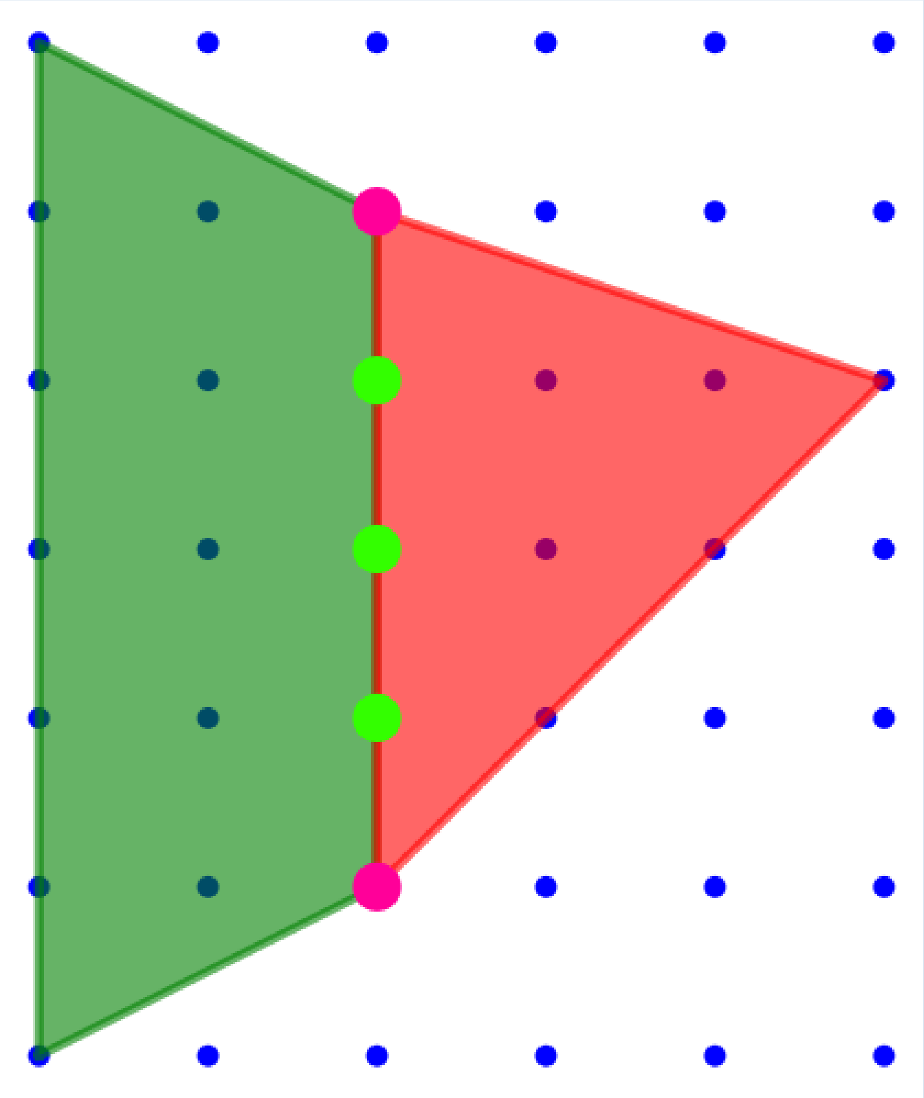
Then

$$C_R = C_P + C_Q + D - 2$$

So

$$C_P + C_Q = C_R - D + 2$$

[ii]



Pick's Theorem

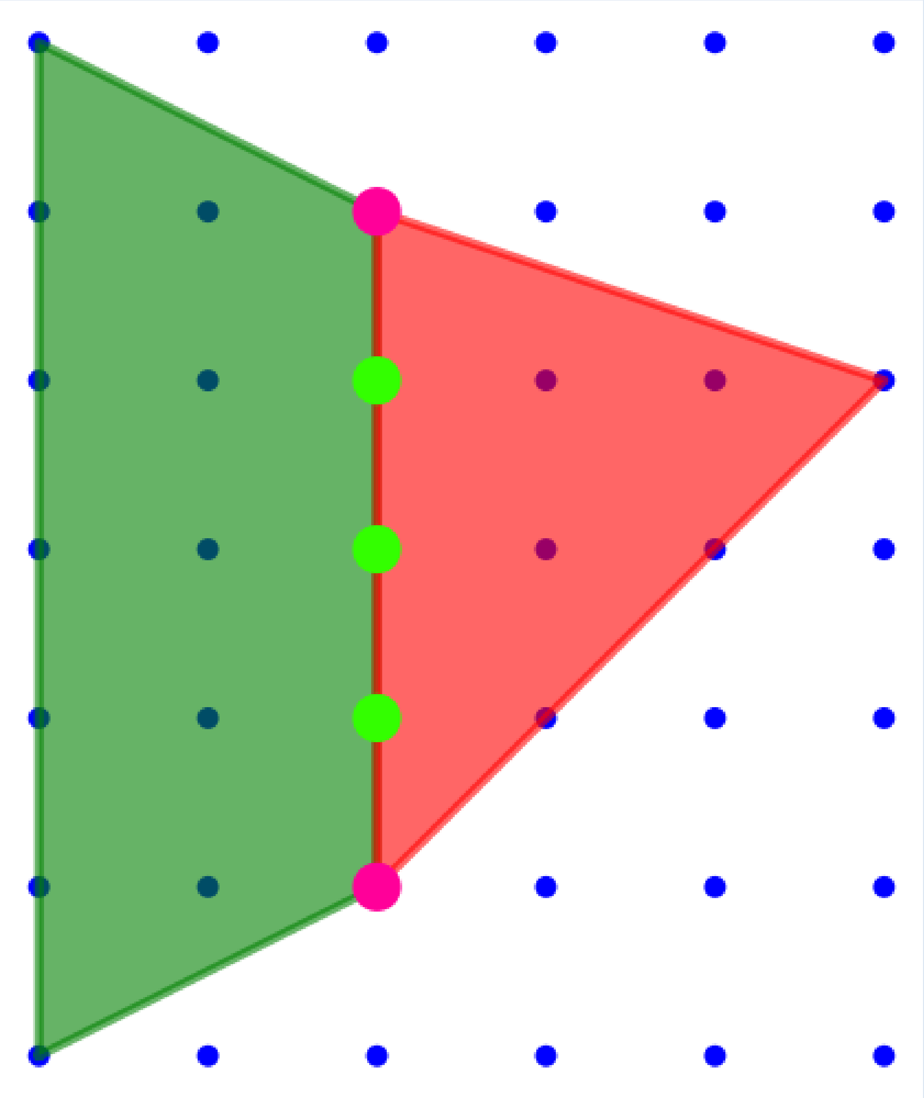
But also…

$$B_R = B_P + B_Q - 2D + 2$$

So

$$B_P + B_Q = B_R + 2D - 2$$

[iii]



Pick's Theorem

Putting these together…

$$A_R = (C_P + C_Q) + \tfrac{1}{2}(B_P + B_Q) - 2 \qquad \text{[i]}$$

becomes

$$A_R = (C_R - D + 2) + \tfrac{1}{2}(B_R + 2D - 2) - 2$$

so

$$A_R = C_R + \tfrac{1}{2}B_R - 1$$

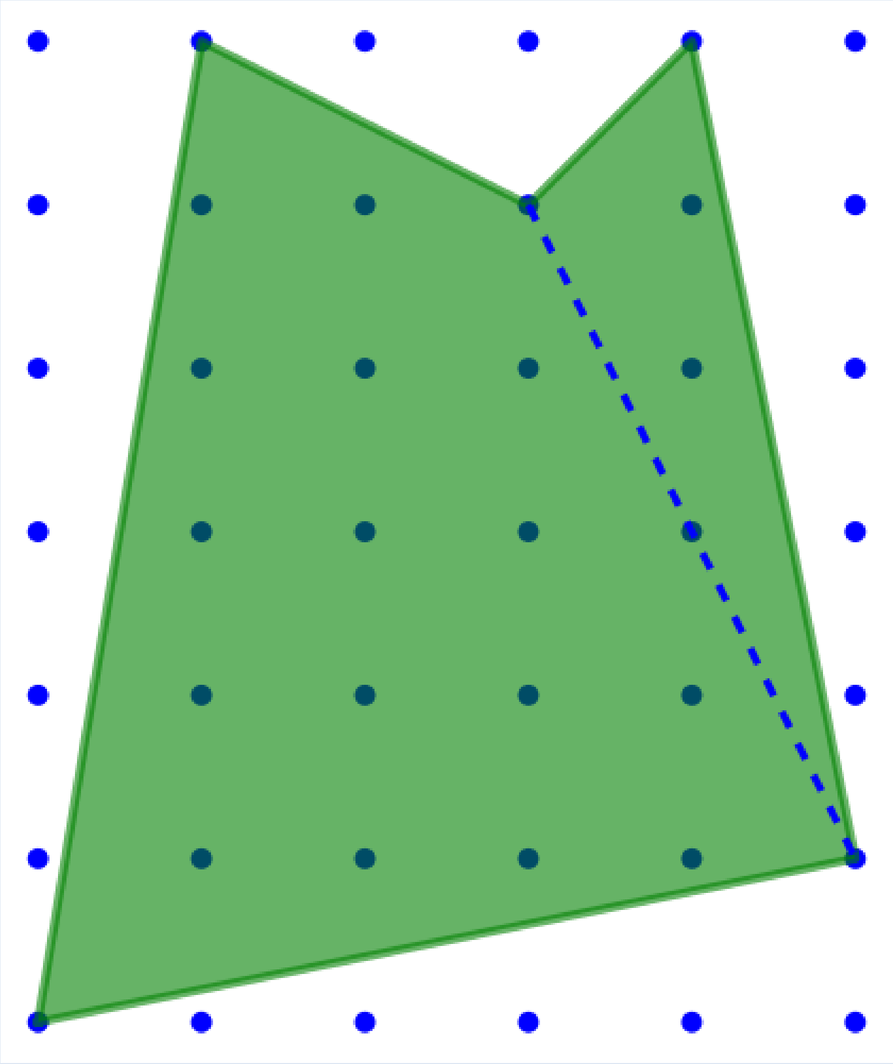
as required.

Pick's Theorem

Step 2

Every lattice polygon can be broken into triangles, joined along edges.

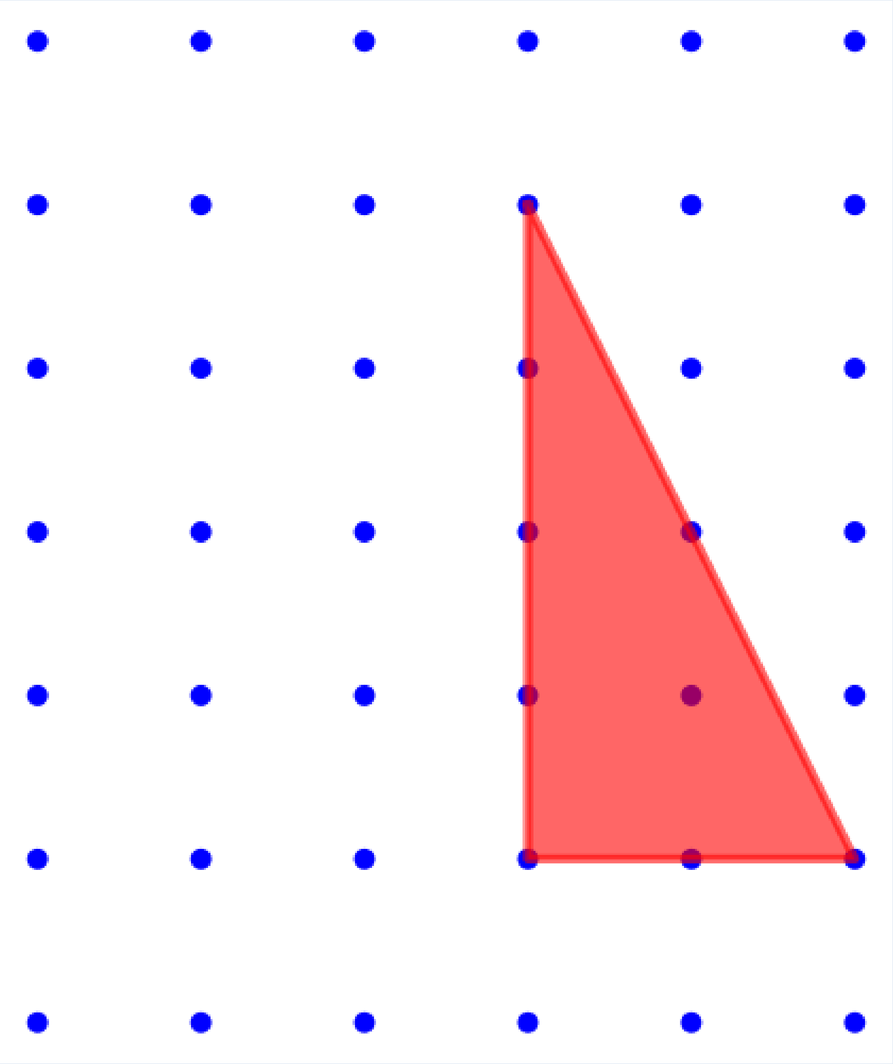
Proof idea:
induction on the number of vertices.



Pick’s Theorem

Step 3 Every lattice triangle satisfies Pick’s theorem.

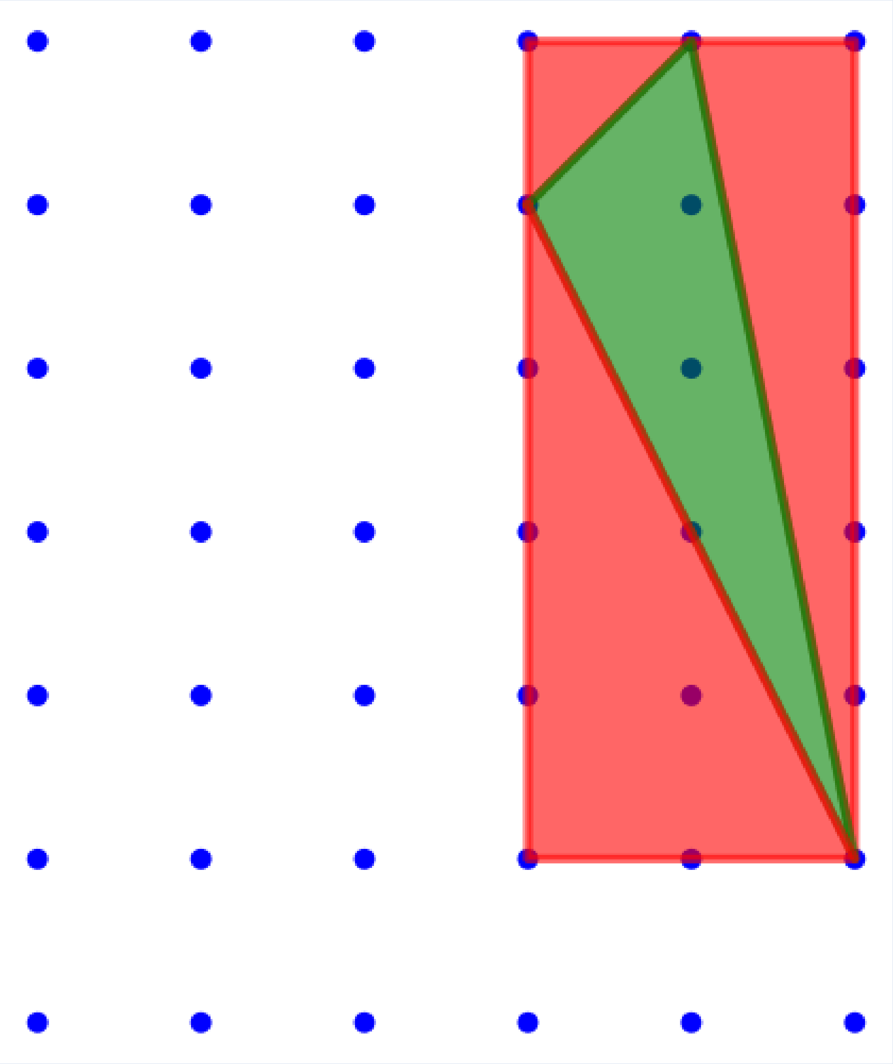
Sketch. (a) It is easy to check that *upright, right-angled* triangles and rectangles (with two edges parallel to the lattice) obey the theorem.



Pick’s Theorem

Sketch (b) Every triangle can be expanded to a regular rectangle by adding on three regular triangles.

By an argument similar to step 1, it follows that the triangle obeys the theorem.



Reeve Tetrahedra

Does Pick's theorem generalise to 3 dimensions?

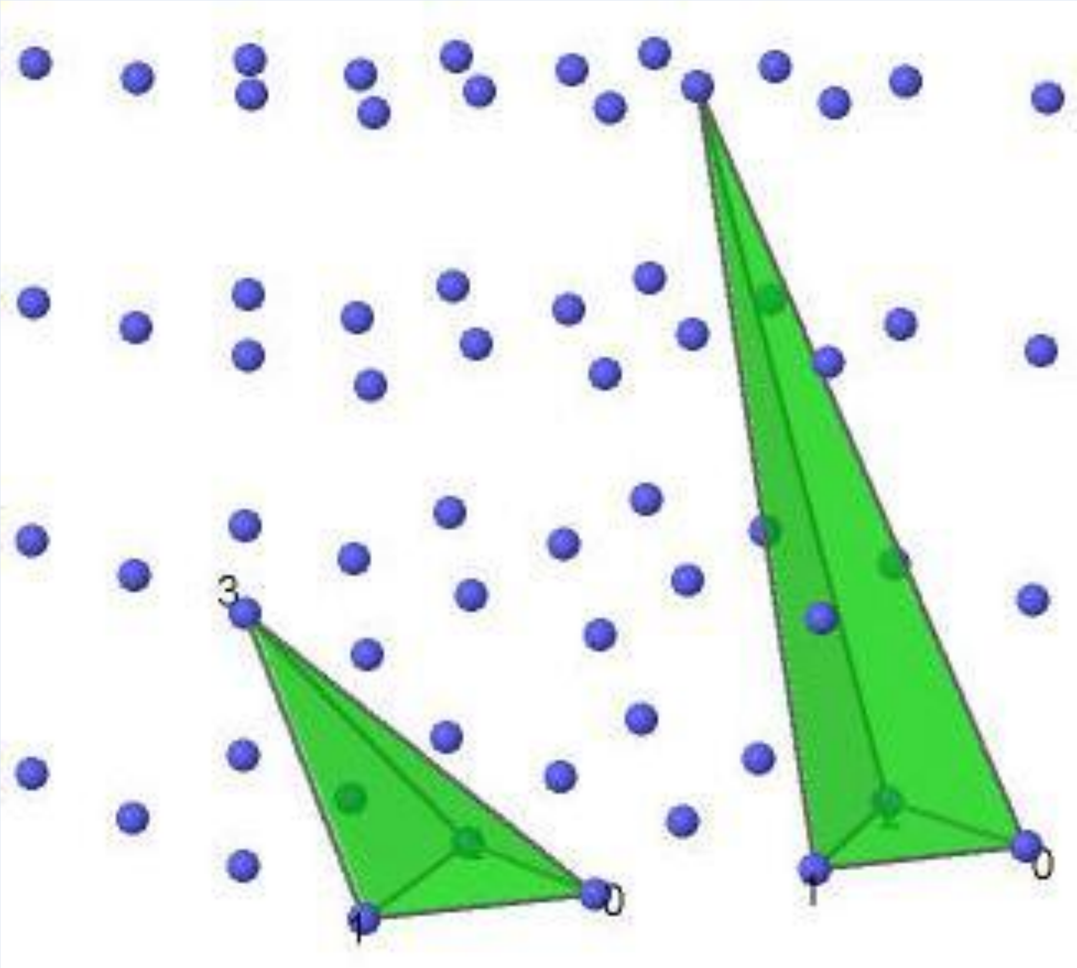
In 1957, John Reeve delivered some bad news.

Reeve tetrahedra have vertices at:

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ \&\ (1,1,r)$$

where r is a positive integer.

Reeve Tetrahedra



All Reeve tetrahedra contain the same number of lattice points (just their four vertices).

But their volumes are *different.*

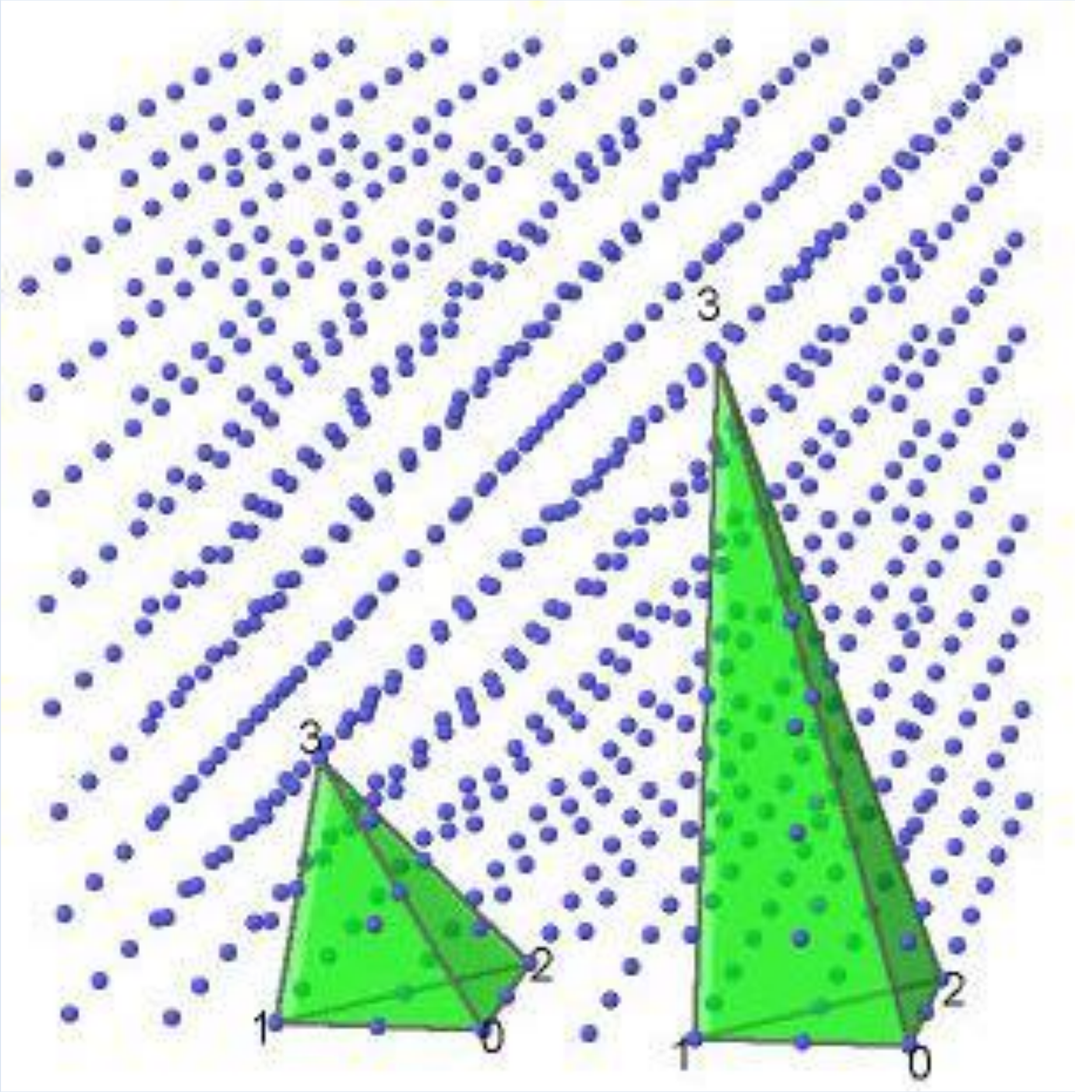
Ehrhart Polynomials

In 1967, Eugène Ehrhart found a way forward.

His idea was to inflate the polyhedron, and then count the number of lattice points in the enlarged shape.

Ehrhart Polynomials

Given a lattice polytope P, and a positive integer n, define nP to the polytope obtained by multiplying the coordinates of every vertex by n.



Ehrhart Polynomials

Define $L_P(n)$ to be the number of lattice points in (or on) *nP*.

Ehrhart proved that $L_P(n)$ is a *polynomial* in *n*.

That is, there are real numbers $a_0,\ldots,a_3$ so that

$$L_P(n) = a_3n^3 + a_2n^2 + a_1n + a_0$$

Ehrhart Polynomials

Examples:

1. If P is a unit cube, nP contains $(n + 1)^3$ points. So,

$$L_P(n) = n^3 + 3n^2 + 3n + 1.$$

2. If P is a Reeve tetrahedron, then

$$L_P(n) = (r/6)n^3 + n^2 + (2 - r/6)n + 1$$

Ehrhart Polynomials

What are $a_0,\ldots,a_3$?

Ehrhart proved that…

- a_3 is the *volume* of P.
- a_2 is the *half the total area* of the faces of P (measured in the induced lattice on each face).
- a_1 is… *???*
- $a_0 = 1$

Ehrhart Polynomials

Generalising to dimension *d* (and allowing holes):

$$L_P(n) = a_d n^d + a_{d-1} n^{d-1} + a_{d-2} n^{d-2} + \ldots + a_1 n + a_0$$

where

- $a_d = V(P)$
- $a_{d-1} = \frac{1}{2} V(\partial P)$ (where ∂P is the boundary of P)
- $a_{d-2}, a_{d-3}, a_{d-4}, \ldots, a_1 = ???$
- $a_0 = \chi(P)$ i.e. the Euler characteristic of P.

Pick's Theorem Revisited

When $d=2$ (and $\chi(P)=1$) we recover Pick's theorem:

$$L_P(n) = V(P)\,n^2 + \tfrac{1}{2}V(\partial P)\,n + \chi(P)$$

Recall that C is the number of lattice points in P's interior, and B is the number on its boundary.

Setting $n=1$, we get $L_P(1) = B+C$, and $V(\partial P) = B$.

So $V(P) = A = C + \tfrac{1}{2}B - 1$, as expected.

Pick's Theorem Revisited

In 2d, we can generalise Pick's theorem to lattice polygons *P* containing *h* holes:

Since $\chi(P) = 1 - h$, we get $A = C + \frac{1}{2}B + (h - 1)$.

Ehrhart Polynomials

What happens if we use negative values of *n?* Does the number $L_P(-n)$ have any geometric meaning?

Answer: the *Ehrhart-MacDonald Reciprocity Law.*

This tells us the number of lattice points in the *interior* ($P°$) of *P*, written as '$L_{P°}$'. It says:

$$L_P(-n) = (-1)^d L_{P°}(n)$$

Ehrhart Polynomials

Two obvious (but hard!) questions:

1. What are a_{d-2}, a_{d-1},…,a_1?

   – They are ***not*** multiples of $V(\partial^2 P)$, $V(\partial^3 P)$,…

   – Reeve tetrahedra illustrate this:

   In all cases $a_1 = 2 - r/6$

   …while $V(\partial^2 P) = 6$ is *independent* of r.

Ehrhart Polynomials

– In 1991, James Pommersheim provided a formula for a_1 when $d=3$, essentially in terms of the angles between faces and number-theoretical objects called *Dedekind sums*.

– In 1994, Sylvain Cappell & Julius Shaneson found formulae for all a_i in terms of related *cotangent expressions*, using deep methods from *Toric Geometry*.

Ehrhart Polynomials

2. What about the *roots* of Ehrhart polynomials?

[Beck, De Loera, Develin, Pfeifle, Stanley, 2004]

- The *real* roots of a convex lattice polytope of dimension *d* all lie in the interval $\left[-d, \left\lfloor \frac{d}{2} \right\rfloor\right)$
- Complex roots are bounded in discs centred at the origin:

d	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Radius	3.6	8.5	15.8	25.7	38.3	53.5	71.4	92.0

Thank You!

References

"Computing the Continuous Discretely"

- Matthias Beck & Sinai Robins [Springer, 2009]

"Enumerative Combinatorics, Volume 1."

- Richard Stanley [CUP, updated 2011]