Knot Maths!

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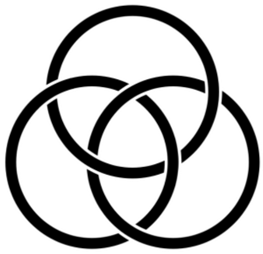
Did you know there is an area of maths which is all about knots? The same sorts of knots we use to tie our shoelaces and neck-ties, and for holding things together. There is just one small difference.

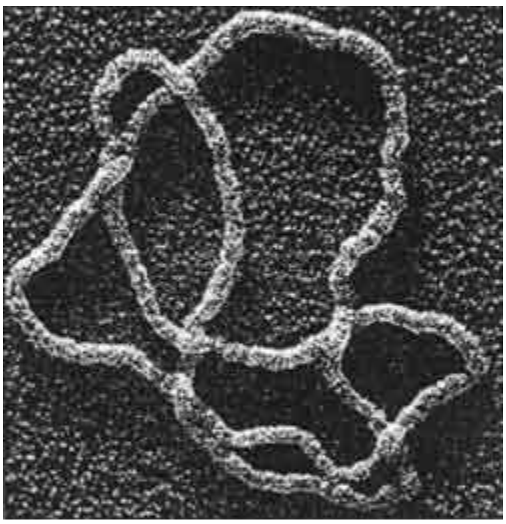
For most people, a knot looks like this:

But a mathematician's knot looks like this:

We always think about knotted *loops*, where the ends of the string have been glued together. Sometimes we will talk about *links*, which are knots which use more than one loop of string, like these *Borromean rings:*



Why are mathematicians interested in knots and links? Because telling knots apart is a *very difficult* question. In fact we still don't know a foolproof way to do it!

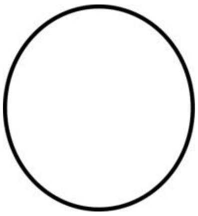


Who cares about knots?

Not just mathematicians and mountaineers! Biologists are also interested in them. This is a picture of some DNA which has got knotted. There are little machines in our cells which are knotting and unknotting our DNA all the time. If the DNA gets too knotted, the cells will die. So understanding knots is a serious business!

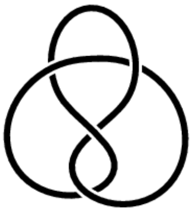
Knots in disguise!

Can you decide which of these knots are really the same?



Unknot

Trefoil knot



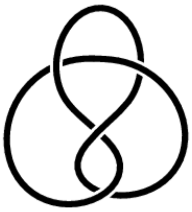
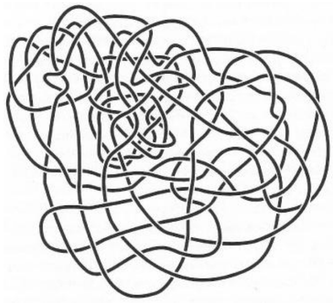


Figure-of-eight knot



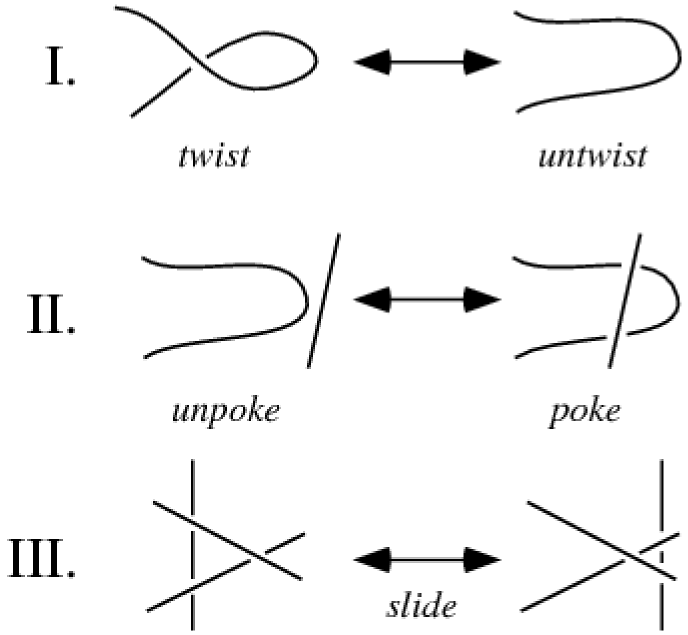
Crossing Number

The *crossing number* of a knot is the number of crossings in a picture of it. Of course, you can draw different pictures of the same knot, as we have just seen. But the crossing number is the *smallest* number of crossings that a picture must have. Knots get more and more complicated as their crossing numbers get bigger and bigger.

The *unknot* has crossing number 0, and is the simplest knot of all. The *trefoil knot* has crossing number 3, and the *figure-of-eight knot* has crossing number 4. There are *two* different knots with crossing number 5, so the crossing number on its own does not determine a knot. In fact, there are 1,701,936 different knots with crossing number up to 16!

Reidemeister moves

In the 1920s, Kurt Reidemeister realised that you could untangle any knot using a picture of it, and just three rules.



This means that if two knots are the same, then we can prove it *only* using Reidemeister’s three rules. For example:

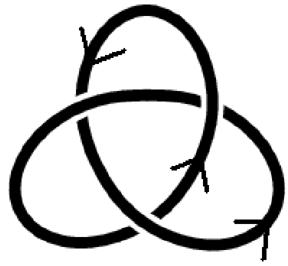
Let’s try untangling some knots and links *only* using Reidemeister moves:

A.

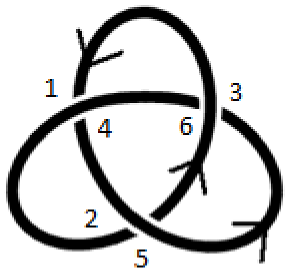
B.

C.

Knot Codes



Apart from drawing a picture of a knot, how can we describe it? One way was invented by Clifford Dowker and Morwen Thistlethwaite. It works like this. First draw a picture of the knot, and then choose a direction to go around it, and draw arrows all around it. (This is called giving the knot an *orientation*.)



Now imagine that you are walking around the knot. The crossings come in two types: *undercrossings* and *overcrossings*. Start just before an undercrossing, and go around the knot, giving each crossing a number until you get back to where you started:

Now each crossing has *two* numbers attached to it. The interesting thing is how they pair up. In this example, we get (1,4) , (3,6) , (5,2).

Notice that each pair is one odd number and one even number. (Usually, even numbers represent *overcrossings*. But occasionally they don't. When an even numbers is an undercrossing, we put a minus sign before its number: so -8 instead of 8. But don't worry about this for today!)

The knot's *code* is the sequence of even numbers which correspond to the odd numbers in order: 1,3,5,... So for the trefoil knot above, it will be "4 6 2".

Try to work out the codes for these knots:

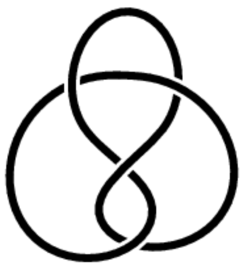
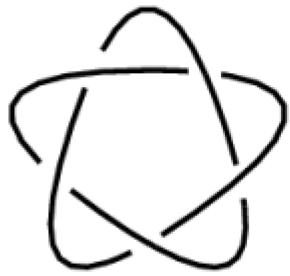


Figure-of-eight knot

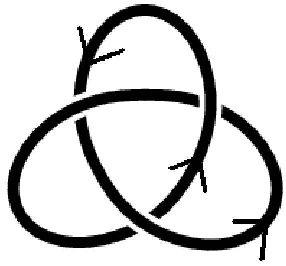


Cinquefoil knot

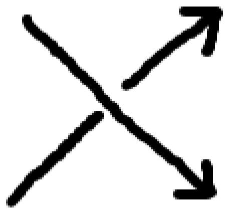
We can also try to go the other way and work out a knot when we are told a code. But sometimes we will fail because not every possible code works. Let's have a go! Try to draw pictures of knots with these codes: "2 4 6" , "2 6 4" , "6 4 2", "2 4 6 8" , "4 6 2 8".

Writhe

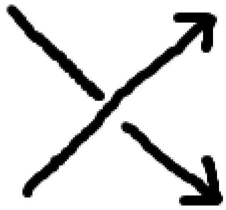
We are going to meet another way to assign a number to a knot. It is called the *writhe*. Again, the first thing to do is to draw arrows on the knot. Make sure all the arrows match up!



Next, we give each crossing a number of either +1 or -1, according to this rule:



+1



-1

How to tell which is which? The rule is: imagine you grab the top strand and want to twist it so that it lines up with the bottom strand. If you need to twist *anticlockwise*, you have a +1 crossing. If you need to twist it *clockwise* it's a -1 crossing.

The final step is to add up the numbers for all the crossings. The total you get is called the *writhe* of the knot. Let's calculate the writhe for the two trefoil knots, and the figure of eight knot.

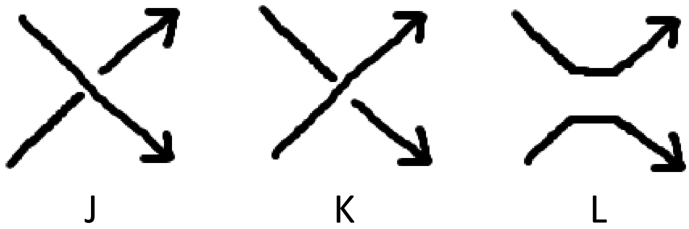
Knot Polynomials

What mathematicians really want is a quick, reliable way to tell whether two knots are different. The writhe *almost* works... but not quite. If you calculate the writhe, then perform some Reidemeister moves and then calculate it again, we want the two answers to come out the same. If you only perform Reidemeister moves types II & III, it will. Unfortunately it doesn't work for type I moves.

But there is a related technique which works for all three moves, discovered by James Alexander in 1928, and improved by John Conway in 1960, and then Vaughan Jones in 1984. We will use Conway's version. This time though, we don't get a number at the end. For every knot, call one K, we get a *polynomial* P_{K}. So it could be $P_{\text{K}} = x^3 - 5x + 3$. This time if two knots have different polynomials (maybe one has $x^3 - 5x + 3$ and the other has $-x^2 - 2$), then the two knots *really are different*. Here are the rules:

Rule 1: the unknot has a simple polynomial, just 1. We write this as $P_{\text{unknot}} = 1$.

Rule 2: imagine we have three oriented knots called J, K, and L, which are identical except at one crossing:



Each of these knots will have a polynomial corresponding to it, let's call them P_J, P_K and P_L. So each of these is an expression with xs and some numbers. The rule says that these three polynomials are related according to this law:

$$P_J - P_K = x \times P_L$$

We can use these rules to build up polynomials for complicated knots, starting with simple ones. For example, $P_{\text{trefoil}} = x^2 + 1$, and $P_{\text{figure-of-eight}} = -x^2 + 1$. So these two knots are *definitely* different.