Groups in supersimple and pseudofinite theories

Richard Elwes   Eric Jaligot   Dugald Macpherson
Mark Ryten

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Abstract. We consider groups G interpretable in a supersimple finite rank theory T such that T^{eq} eliminates $\exists^{\infty}$. It is shown that G has a definable soluble radical. If G has rank 2, then if G is pseudofinite it is soluble-by-finite, and partial results are obtained under weaker hypotheses, such as unimodularity of the theory. A classification is obtained when T is pseudofinite and G has a definable and definably primitive action on a rank 1 set.

1 Introduction

Shelah's notion of *simple* first order theory provides a context, broader than that of stable theories, where there is an abstract model-theoretic notion of independence. The initial examples of simple (in fact, supersimple) theories include the random graph, smoothly approximable structures, and pseudofinite fields. These all arise from finite structures, and indeed have the finite model property. Our goal is to develop the structure theory for *groups* with supersimple theory, often with extra hypotheses. One such extra assumption is *pseudofiniteness*, which seems very natural in view of the wealth of examples.

The theory of groups definable in supersimple (or just simple) theories is not nearly as well developed as under stability assumptions. The main source is [42]. There are results on connected components over specific parameter sets, and a theory of generic types (but without stationarity). There is a version of the Zilber Indecomposability Theorem, with some of the expected consequences. Much of the theory in [42] is developed in the more general context of hyperdefinable groups. But issues settled early for *superstable* groups are wide open in the supersimple context. It is not known if there could be a non-abelian simple group of SU-rank 1, or if every infinite group definable in a finite rank supersimple theory contains an infinite abelian subgroup. Strong results have been obtained for groups definable in particular classes of supersimple theories, such as theories of smoothly approximable structures [28, 11], and of pseudofinite fields [26]. These theories are *measurable*, in the sense of [34], and there are the beginnings of a theory of groups definable in measurable theories; see for example [34], [19], [20], [39].

In this paper we investigate groups definable in supersimple finite rank theories, often working under the extra assumption that the theory of the group is *pseudofinite*, that is, it has infinite models but every sentence in it has a finite model. Part of the motivation is that every pseudofinite group which is simple (in the group-theoretic sense) has, as a pure group, supersimple finite rank theory. This, partly stimulated also by the programme on simple groups of finite Morley rank, makes it natural to try to approach finite groups from model-theoretic hypotheses, e.g. to recover families of finite simple groups, along with their representations. The paper [31] also contains material in this direction. The present paper builds on results in [20] and [39].

We work under various assumptions, and with various classes of theories. When we say that a theory T^{eq} eliminates $\exists^\infty$, we mean that for any uniformly definable family of sets *in the sense of* M^{eq}, there is an upper bound on the size of its finite members. The broadest class we consider is the class $\mathcal{S}$ of all structures of finite rank interpretable in some supersimple theory T such that T^{eq} eliminates the quantifier $\exists^\infty$. The rank function considered here is *SU*-rank, and the rank of a definable set X will be denoted by $\mathrm{rk}(X)$. So far as possible, we work just with the assumptions for the class $\mathcal{S}$, but sometimes we work with the class $\mathcal{M}$ of members of $\mathcal{S}$ which are also 'unimodular'. Here, a structure M is *unimodular* [25] if, whenever X, Y are definable subsets of M^{eq} and $f : X \to Y$ and $g : X \to Y$ are definable surjections such that f is everywhere k-to-1 and g is everywhere l-to-1 ($k, l \in \mathbb{N}$), we have $k = l$. It is easily checked that unimodularity is a property of the *theory* of M, that every pseudofinite theory is unimodular, and that every measurable theory is unimodular; see e.g. Section 3 of [19]. Sometimes, we strengthen 'unimodular' to the assumption 'pseudofinite', to obtain the class $\mathcal{F}$ of pseudofinite members of $\mathcal{S}$; so $\mathcal{F} \subset \mathcal{M} \subset \mathcal{S}$. The class $\mathcal{F}$ properly contains another class of structures introduced in [34] and [18], and occasionally mentioned below; namely, the class of structures elementarily equivalent to a non-principal ultraproduct of an *asymptotic class* of finite structures. When investigating groups in $\mathcal{F}$ we try, so far as possible, to work without the classification of finite simple groups (CFSG).

Our main results are the following.

Theorem 1.1 *Let $G \in \mathcal{S}$. Then*

(i) any soluble subgroup of G normalised by $H \leq G$ is contained in a definable H-invariant soluble subgroup of G;

(ii) G has a largest soluble normal subgroup $R(G)$, and $R(G)$ is definable.

Theorem 1.2 *Let $G \in \mathcal{F}$ have rank two. Then G is soluble-by-finite.*

Theorem 1.3 *Let $(X, G) \in \mathcal{F}$ be a definably primitive permutation group, and suppose that* $\mathrm{rk}(X) = 1$. *Let* $S = \mathrm{Soc}(G)$. *Then one of the following holds.*

(i) $\mathrm{rk}(G) = 1$, *and S is divisible torsion-free abelian or elementary abelian, has finite index in G, and acts regularly on X.*

(ii) $\mathrm{rk}(G) = 2$. *Here S is abelian so regular and identified with X. There is an interpretable pseudofinite field F with additive group X, and $G \leq \mathrm{AGL}_1(F)$ (a subgroup of finite index), in the natural action.*

(iii) $\mathrm{rk}(G) = 3$. *There is an interpretable pseudofinite field F, $S = \mathrm{PSL}_2(F)$, $\mathrm{PSL}_2(F) \leq G \leq \mathrm{P\Gamma L}_2(F)$, and G has the natural action on $\mathrm{PG}_1(F)$.*

The classification of finite simple groups is used in the proof of Theorem 1.3, but not in the proof of 1.2.

We give in Section 2 some background results, mostly known, which are used repeatedly in the paper. In Section 3 we consider soluble groups in $\mathcal{S}$. If G is a stable group, then every soluble subgroup of G is contained in a definable soluble subgroup of G of the same derived length, and every nilpotent subgroup of G is contained in a definable nilpotent one of the same class; see for example [43, Theorems 1.1.10, 1.2.11]. In Section 3 we obtain partial analogues (at least in the soluble case) for groups in $\mathcal{S}$, in particular Theorem 1.1. It follows from Theorem 1.1(ii) that there is a good general description of groups in $\mathcal{F}$ (see e.g. Proposition 3.4), and the fine structure of such groups will (assuming CFSG) reduce to understanding soluble groups in $\mathcal{F}$.

We then in Section 4 investigate rank 2 groups in $\mathcal{M}$. The aim is to show that such groups are soluble-by-finite. This would be an analogue of results in the superstable case [8], the o-minimal case [36], and, more technically, of the description of thorn-U rank two super-rosy NIP groups with finitely satisfiable generics [16]. We have not managed this, but give partial results (Theorem 1.2). This result was proved in [20], under the strong additional assumption that the group is an ultraproduct of members of an 'asymptotic class'. Unlike that of [20], the proof given here does not use CFSG.

In Section 5 we consider *permutation* groups (X, G) in $\mathcal{F}$, that is structures (X, G) where G is a group with a faithful definable action on X. Recall that the transitive permutation group (X, G) is *primitive* if there is no proper non-trivial G-invariant equivalence relation on X, or equivalently, if point stabilisers G_x (for $x \in X$) are maximal subgroups of G. We work under the assumption that $\mathrm{rk}(X) = 1$, and that G is *definably primitive* on X, that is, there is no proper non-trivial *definable* G-invariant equivalence relation on X. Using CFSG, we prove Theorem 1.3, the expected analogue of Hrushovski's theorem about groups in a stable theory acting definably and transitively on a strongly minimal set. That is, we show that $\mathrm{rk}(G) \leq 3$, and that the expected classification holds. The paper concludes with a short section related to the third and fourth questions below. Using CFSG, we note that any rank 3 simple group in $\mathcal{F}$ is isomorphic to $\mathrm{PSL}_2(K)$ for some pseudofinite field K. Then, we note that definably primitive permutation groups in $\mathcal{F}$ arise from finite permutation groups (X, G) such that $|G|$ is polynomially bounded in terms of $|X|$.

There are several easily formulated further questions.

1. If $G \in \mathcal{S}$ (possibly with extra assumptions, such as unimodularity), then is the product of the nilpotent normal subgroups of G necessarily nilpotent and definable? That is, is there a good notion of *Fitting subgroup*?

2. Is every rank 2 group in $\mathcal{M}$ soluble-by-finite?

3. Describe rank three simple groups in $\mathcal{F}$ without use of CFSG.

4. Show that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that if (X, G) is a definably primitive permutation group in $\mathcal{F}$ then $\text{rk}(G) \leq f(\text{rk}(X))$.

Regarding Question 4, the analogous result is proved in the finite Morley rank case in [5], and in the o-minimal case in [33]. Using results from [31] it may be straightforward – see Remark 6.3.

Notation, and conventions. In simple theories, there are various familiar notions of rank, namely D-rank, SU-rank, and $S1$-rank. It is shown in [29] that in a supersimple theory, if a definable set X has finite rank in any of these senses, then it has the same rank in each sense. Here we just write it as $\text{rk}(X)$, except in a few possibly infinite rank situations where we specify that rank is SU-rank. (We remark that for *types*, even in a finite rank situation, these ranks may differ – see [42, 5.1.15].) In addition, in a measurable theory, there is a notion of *dimension* of a definable set. The dimension is an upper bound on the SU-rank, but they may not coincide. However, this dimension behaves essentially like rank, and in a measurable theory the results of this paper all hold if rank is replaced by dimension.

We denote by (X, G) a permutation group G on a set X. Its *degree* is $|X|$. If $x \in X$ then G_x denotes the stabiliser of x. A G-*congruence* on X is a G-invariant equivalence relation on X. We write C_n for the cyclic group of order n. The *socle* of a group G, denoted $\text{Soc}(G)$, is the subgroup generated by the minimal normal subgroups of G. If x, y are elements of the group G then x^y denotes yxy^{-1} (rather than $y^{-1}xy$), and if $H \leq G$ then $H^x := \{h^x : h \in H\}$.

We tend to use the words 'definable' and 'interpretable' interchangeably, allowing quotients. When considering ultraproducts of members of $\mathcal{F}$, we typically consider an infinite family $\{M_j : j \in J\}$ of finite structures, and a non-principal ultrafilter on J (usually with no specific notation). We say that some property P holds for *ultrafilter-many* $j \in J$ if $\{j \in J : M_j \text{ satisfies } P\}$ lies in the ultrafilter.

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2 Background Results

We list here some tools which will be heavily used in the paper.

First, an easy consequence of the Lascar Inequalities for SU-rank, and our finiteness of rank assumption is the following. Below, G/H refers just to the coset space, a definable set in the sense of T^{eq}.

Lemma 2.1 *[42, p. 168] (i) If G is a group with supersimple theory, and $H \leq G$, then $|G : H|$ is finite if and only if $SU(H) = SU(G)$.*

(ii) Let $G \in \mathcal{S}$, and H be a definable subgroup of G. Then $\mathrm{rk}(H)+\mathrm{rk}(G/H) = \mathrm{rk}(G)$.

Next, a version of the Zilber Indecomposability Theorem relevant to supersimple theories was first proved around 1991 in some notes of Hrushovski, which led to the paper [24]. Hrushovski assumed that the underlying theory is an $S1$-theory, which includes the assumption that rank is definable, but this is not needed for Theorem 2.2 below. A more general statement was proved by Wagner [42, Theorem 5.4.5] for hyperdefinable groups in simple theories. The consequence we use is the following [20, Remark 3.5].

Theorem 2.2 (Indecomposability Theorem) *Let G be a group in $\mathcal{S}$, and let $\{X_i : i \in I\}$ be a collection of definable subsets of G. Then there exists a definable subgroup H of G such that:*

(i) $H \leq \langle X_i : i \in I\rangle$, and there are $n \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, and $i_1, \ldots, i_n \in I$, such that $H \leq X_{i_1}^{\varepsilon_1} \ldots X_{i_n}^{\varepsilon_n}$.

(ii) X_i/H is finite for each $i \in I$.

If the collection of X_i is setwise invariant under some group Σ of definable automorphisms of G, then H may be chosen to be Σ-invariant.

We mention some consequences of the Indecomposability Theorem used here. A definable group G is *definably simple* if it has no proper non-trivial definable normal subgroups. The proofs in [24] are for $S1$-theories, but just use the Indecomposability Theorem as stated above.

Corollary 2.3 *[24, Corollary 7.4], [42, Proposition 5.4.9] If G is a non-abelian definably simple group in $\mathcal{S}$, then G is a simple group.*

Corollary 2.4 *[24, Corollary 7.1] If G is a group in $\mathcal{S}$, then the derived subgroup G' is definable.*

We occasionally use the following theorem of Schlichting [40], reproved by Bergman and Lenstra [3], with the formulation below as in [42, Theorem 4.2.4]. If G is a group, subgroups H, K of G are *commensurable* if both of $|H : H \cap K|$ and $|K : H \cap K|$ are finite. A family $\mathcal{H}$ of subgroups of G is *uniformly commensurable* if there is $n \in \mathbb{N}$ such that $|H_1 : H_1 \cap H_2| < n$ for all $H_1, H_2 \in \mathcal{H}$. Likewise $K \leq G$ is *uniformly commensurable* to $\mathcal{H}$ if there is some $n \in \mathbb{N}$ such that for all $H \in \mathcal{H}$, we have $|H : H \cap K| < n$ and $|K : H \cap K| < n$.

Theorem 2.5 *Let G be a group, and $\mathcal{H}$ be a uniformly commensurable family of subgroups of G. Then there is a subgroup N of G which is uniformly commensurable to all members of $\mathcal{H}$ and is invariant under all automorphisms of G which fix $\mathcal{H}$ setwise. The group N is a finite extension of the intersection of finitely many elements of $\mathcal{H}$.*

In conjunction with 2.5 we make occasional use of the following easy lemma.

Lemma 2.6 *Let (X, G) be a transitive permutation group definable in a structure whose theory eliminates $\exists^\infty$ and does not have the strict order property. Define $\sim$ on X by putting $x \sim y$ (for $x, y \in X$) if and only if $|G_x : G_{xy}| < \infty$. Then $\sim$ is a definable G-congruence.*

Proof. A proof assuming measurability is given in [20, Proposition 6.1]. Measure was used just to prove symmetry, but if $\sim$ was not symmetric it would be a definable partial order on G with an infinite chain, yielding the strict order property. □

The last lemma and the Indecomposability Theorem yield the following. The assumptions in the proof in [20] include measurability. This is not needed, since the only use is through the last lemma. Below, we say that a subgroup K of a group G is *uniformly maximal* in G if there is a positive integer n such that for any $g \in G \setminus K$ and $h \in G$, h is equal to a word of length at most n of the form $h = k_1 g^{\varepsilon_1} k_2 g^{\varepsilon_2} n_3 \ldots k_{t-1} g^{\varepsilon_{t-1}} k_t$, where $k_i \in K$ for each i and $\varepsilon_i \in \{1, -1\}$. If K is definable is G, this ensures that maximality is first-order expressible, and preserved under elementary extensions.

Lemma 2.7 *[20] Let (X, G) be a definable primitive permutation group in $\mathcal{S}$, with $\mathrm{rk}(G) > \mathrm{rk}(X)$. Then (X, G) is primitive, and for $x \in X$ the stabiliser G_x is uniformly maximal in G.*

Proof. See [20, Theorem 6.2]. For the last assertion, note that if G_x were not uniformly maximal in G then there would be an ω-saturated elementary extension (X^*, G^*) of (X, G) which is definably primitive but not primitive, contrary to the first assertion. □

The Zilber Field Interpretation Theorem for groups of finite Morley rank has the following analogue in our context. If a definable group H acts definably on a definable group A, we say that A is *H-minimal* if there are no proper non-trivial definable H-invariant subgroups of A.

Proposition 2.8 *[20, Proposition 3.6] (Field Interpretation Theorem) Let $G = AH$ be a group in $\mathcal{S}$. Suppose that A and H are each abelian and definable and H is infinite and normalises A, that A is H-minimal, and that $C_H(A) = \{1\}$. Then the following hold.*

(i) The subring $K = \mathbb{Z}[H]/\mathrm{ann}_{\mathbb{Z}[H]}(A)$ of $\mathrm{End}(A)$ is a definable field; in fact, there is an integer l such that every element of K can be represented as an endomorphism $\sum_{i=1}^{l} (-1)^{\varepsilon_i} h_i \ \ (h_i \in H)$, where $\varepsilon_i \in \{1, -1\}$.

(ii) A is definably isomorphic to K^+, H is isomorphic to a subgroup J of K^* and the action by conjugation of H on A is (after identification of K^+ with A) its action by multiplication on K.

Proof. This is essentially proved in [20, Proposition 3.6], but since it is assumed there that $\mathrm{rk}(H) = \mathrm{rk}(A) = 1$, and since there are inaccuracies in the proof in [20], we give the details. We write A additively.

Let B be the union of the finite H-orbits on A. Then B is an H-invariant subgroup of A, and is definable as $\exists^\infty$ is definable. Hence $B = \{0\}$ or $B = A$, by H-minimality.

We claim that $B = \{0\}$, so suppose for a contradiction that $B = A$. Let $a \in A \setminus \{0\}$ and $h \in C_H(a) \setminus \{1\}$; such h exists, as $|H : C_H(a)|$ is finite (as a^H is by assumption finite) and H is infinite. Then $C_A(h)$ is a non-trivial definable subgroup of A, and is H-invariant as H is abelian. The H-minimality of A forces then $C_A(h) = A$, a contradiction as H acts faithfully on A. Hence $B = \{0\}$.

(i) The group ring $\mathbb{Z}H$ can be viewed as a ring of endomorphisms of A, extending the H-action by conjugation. If $r \in \mathbb{Z}H$ (or lies in a quotient which acts on A) and $a \in A$, we write $r \cdot a$ for the image of a under r. If $r \in \mathbb{Z}H$ then as H is abelian, $\mathrm{Ker}(r)$ and $\mathrm{Im}(r)$ are definable H-invariant subgroups of A. Thus either $\mathrm{Ker}(r) = A$, in which case $r \in \mathrm{ann}_{\mathbb{Z}H}(A)$, or $\mathrm{Ker}(r) = \{0\}$, and $\mathrm{Im}(r) = A$. In particular, if $R = \mathbb{Z}H/\mathrm{ann}_{\mathbb{Z}H}(A)$, then non-zero elements of R act as automorphisms of A.

Let $a \in A \setminus \{0\}$, and let W be the orbit of a under the action of H. Then W is infinite and H-invariant. Thus, by Theorem 2.2, there is an infinite definable H-invariant subgroup C of A, with $C \leq \langle W \rangle$. By H-minimality, we have $C = A$. Furthermore, again by Theorem 2.2, for some ℓ we have $A = (-1)^{\varepsilon_1} W + \ldots + (-1)^{\varepsilon_\ell} W$, where $\varepsilon_1, \ldots \varepsilon_\ell \in \{1, -1\}$.

Define $K = \{\Sigma_{i=1}^{\ell} (-1)^{\varepsilon_i} h_i : h \in H\}/\mathrm{ann}_{\mathbb{Z}[H]}(A)$. Then $K \subseteq R \subseteq \mathrm{End}(A)$.

Suppose $r \in R \setminus \{0\}$. Then r induced an automorphism of A, so there is $b \in A$ with $r \cdot b = a$. Since $b \in (-1)^{\varepsilon_1} W + \ldots + (-1)^{\varepsilon_\ell} W$, by construction of K there is $s \in K$ with $sa = b$. Thus, $(rs - 1) \cdot a = 0$ so as non-zero elements of R are automorphisms, $rs = 1$. Hence R is a field. In addition, we have shown that every non-zero element of R has an inverse in K; in particular, $s^{-1} = r \in K$, so $R = K$.

(ii) Define $i_a : K \to A$ by $i_a(r) = r \cdot a$. Then (by the definition of K), i_a is an additive isomorphism $K \to A$. Define a multiplication $\otimes$ on A as follows. If $b = r \cdot a$ and $c = s \cdot a$, put $b \otimes c := i_a(rs)$. Then i_a is an isomorphism of fields sending H to a subgroup of $(A, \otimes)$. □

Recall that a *BFC group* is a group G such that for some $n \in \mathbb{N}$, all conjugacy classes in G have size at most n. We shall use the following theorem.

Theorem 2.9 *[37, Theorem 3.1] Let G be a BFC group. Then*
(i) the derived subgroup G' is finite,
(ii) G has a definable characteristic subgroup H of finite index such that H' is a finite subgroup of $Z(H)$.

Proof. For (ii), put $H := C_G(G')$.

Corollary 2.10 *Let G be a group such that* $\mathrm{Th}(G)$ *eliminates $\exists^\infty$. Then G has a definable characteristic subgroup N, consisting of the finite conjugacy classes of G, such that N' is finite.*

Proof. The set of conjugacy classes is a uniformly definable family of sets, so there is an upper bound n on the sizes of the finite ones. The union of the finite conjugacy classes of G is clearly a characteristic subgroup, and is definable. It is a BFC group, so Theorem 2.9 is applicable. □

Together with counting arguments, Theorem 2.9 was used in [20] (see also [34, Theorem 5.1.5]) to prove the following.

Theorem 2.11 *Let $G \in \mathcal{M}$, with $\mathrm{rk}(G) = 1$. Then G is finite-by-abelian-by-finite. More precisely, G has a definable finite index characteristic subgroup N, contained in the union of the finite conjugacy classes of G, such that $N' \leq Z(N)$ is finite.*

The proof of this yields also the following, noted in [19, Lemma 7.4].

Proposition 2.12 *Let $G \in \mathcal{M}$. Then some $g \in G \setminus \{1\}$ has infinite centraliser.*

Lemma 2.13 *Let p be a prime, and let $A \in \mathcal{M}$ be abelian, and have no p-torsion. Then A is (uniquely) p-divisible.*

Proof. Let $f : A \to A$ be the group homomorphism $x \mapsto px$. Since A has no p-torsion, f is injective, so $B := f(A)$ has the same rank as A, so has finite index in A by Lemma 2.1. For p-divisibility, it suffices to show that $A = B$. As f is injective, each element of A will then be *uniquely* p-divisible.

Suppose $|A : B| = n$, and write $A = Bx_1 \cup \ldots \cup Bx_n$ (a disjoint union), where $x_1 = 1$. Define $g : A \to A$ by putting $g(y) = f^{-1}(yx_i^{-1})$ for $y \in Bx_i$ (for $i = 1, \ldots, n$). Then g is a definable n-to-1 surjection $A \to A$. Since the identity map $A \to A$ is 1-to-1, it follows by unimodularity that $n = 1$. □

We record below a result of J.S. Wilson [44], slightly strengthened through results in the PhD thesis of Ryten.

Proposition 2.14 *Let G be a simple pseudofinite group. Then*
(i) G is a group of Lie type, possibly twisted, over a pseudofinite field,
(ii) the theory of G (as a pure group) is measurable.

Proof. (i) By the main theorem of [44], G is elementarily equivalent to an ultraproduct $\Pi G(q)/\mathcal{U}$ of finite groups $G(q)$ of a fixed Lie type over finite fields $\mathbb{F}_q$. Furthermore, by the results in Chapter 5 of [39], the groups $G(q)$ are uniformly bi-interpretable over parameters with $\mathbb{F}_q$ (or, in the case of Suzuki and Ree groups, with structures $(\mathbb{F}_q, \sigma)$ for an appropriate field automorphism σ). In particular, the theory of G states that there is a field F (or difference field (F, σ)) and a group of Lie type over F which is definably (in G) isomorphic to G. Part (i) follows.

(ii) Theorem 1.1.1 of [39] states that any family of finite simple groups of fixed Lie type is an asymptotic class (a notion introduced in full generality in [18]). It follows, e.g. by [17, 2.1.10], that any non-principal ultraproduct of such a family is measurable. □

Remark 2.15 In the contexts of this paper, a useful example to bear in mind is that of extraspecial groups. Let p be an odd prime. A p-group P is *extraspecial* if $P' = Z(P) \cong C_p$ and $P/Z(P)$ is elementary abelian. Extraspecial p-groups for odd p have exponent p or p^2. A finite extraspecial p-group of exponent p has order p^{2m+1} for some $m > 0$, and is determined up to isomorphism by its order: it is a central product of extraspecial groups of order p^3. It was shown in [21] that extraspecial p-groups of exponent p are $\aleph_0$-categorical. In fact, they are smoothly approximable, and also measurable of rank 1, since the class of finite extraspecial p-groups of exponent p (a fixed prime) is a one-dimensional asymptotic class (see e.g. [34, Proposition 3.11]).

If P is an extraspecial p-group of (odd) exponent p, then $P/Z(P)$ has the structure of a vector space over $\mathbb{F}_p$ (identified with $Z(P)$), and the commutator map to the centre endows it with a symplectic bilinear form. In particular, if P is countable then (viewed additively) $P/Z(P)$ has a basis over $\mathbb{F}_p$ of the form $\{e_i, f_i : i \in I\}$ such that $[e_i, f_j] = \delta_{ij} \in \mathbb{F}_p$ (Kronecker delta) and $[e_i, e_j] = [f_i, f_j] = 0$. If P is countably infinite, it does not have the descending chain condition on centralisers; for putting $I = \omega$, one has a strict descending chain $P > C_P(e_1, f_1) > C_P(e_1, e_2, f_1, f_2) > \ldots$ with intersection $Z(P)$. Thus, centralisers of elements all have finite index in P, and their intersection is $Z(P)$; in particular, there is no smallest definable subgroup of 'bounded' index.

3 Soluble subgroups

In this section we obtain basic results about soluble subgroups of groups in $\mathcal{S}$, in particular proving Theorem 1.1 (Propositions 3.2 and 3.3). Note that the first two results below, and hence also Theorem 1.1(i), do not require the finiteness of rank assumption.

Lemma 3.1 *Let G be an infinite group definable in a supersimple theory T such that T^{eq} eliminates $\exists^\infty$.*

(i) Let $H \leq G$ be an infinite finite-by-abelian subgroup. Then H is contained in an infinite definable finite-by-abelian subgroup $K \leq G$.

(ii) If H is in addition normalised by $B \leq G$, then K may be chosen to be normalised by B.

Proof. (i) Suppose $F \lhd H$ is finite and H/F is abelian. Now let J be a definable subgroup of G of minimal SU-rank such that $H \leq J$ and $F \lhd J$. Such J exists, as $N_G(F)$ is definable and contains H. For $h \in H$, put $\bar{h} := hF \in J/F$. Let $C_J(\bar{h})$ be the stabiliser of $\bar{h}$ in the action of J by conjugation on J/F. So $C_J(\bar{h}) \leq J$. Also H/F is abelian, so $H \leq C_J(\bar{h})$. As J has minimal rank, $|J : C_J(\bar{h})| < \infty$. Of course $C_J(h) \subseteq C_J(\bar{h})$, and as F is finite, $|C_J(\bar{h}) : C_J(h)| < \infty$. It follows that $|J : C_J(h)| < \infty$, and so h^J is finite.

Now let $L := \{j \in J : |j^J| < \infty\}$. Then as $\exists^\infty$ is eliminable, L is definable, with $H \leq L$, and L is a BFC group, so by Theorem 2.9, L' is finite.

(ii) Let J be of minimal SU-rank among definable subgroups of G which are BFC and contain H; such J exists, by (i). For $b \in B$ let $J^b := bJb^{-1}$.

Then $\mathcal{J} := \{J^b : b \in B\}$ is a family of definable supergroups of H, and for all $b, b' \in B$ we have $|J^b : J^b \cap J^{b'}| < \infty$, by the minimality of $\mathrm{SU}(J)$. Moreover as $\exists^\infty$ is eliminated, this index is bounded, and $\mathcal{J}$ is a uniformly commensurable family. So by Theorem 2.5, there is a subgroup K_0 of G which is uniformly commensurable to $\mathcal{J}$, and which is normalised by B. The group K_0 is a finite extension of some group $J^{b_1} \cap \ldots \cap J^{b_r}$, where $b_1, \ldots, b_r \in B$. In particular, K_0 is definable, and contains H. Let K be the union of the finite conjugacy classes of K_0. Then K is definable. Since each J^{b_i} is BFC, so is $J^{b_1} \cap \ldots \cap J^{b_r}$, and hence $J^{b_1} \cap \ldots \cap J^{b_r} \leq K$. Thus, as $H \leq J$ and B normalises H, $H \leq K$. As $K \operatorname{char} K_0$, K is normalised by B. Finally, as K is a BFC group, it is finite-by-abelian. □

Proposition 3.2 *(i) Let G be an infinite group definable in a supersimple theory T such that T^{eq} eliminates $\exists^\infty$, and let S be a soluble subgroup of G. Then S is contained in a definable soluble subgroup H of G.*

(ii) If S is also normalised by $B \leq G$, then H can be chosen to be normalised by B.

Proof. (i) We argue by induction on the derived length d of S. Let $S^{(r)}$ be the smallest infinite term in the derived series of S, so $S^{(r+1)}$ is finite. We work in $R := N_G(S^{(r+1)})$, a definable subgroup of G containing S.

The group $S^{(r)}$ is a finite-by-abelian subgroup of R which is normalised by S, so by Lemma 3.1(ii), it is contained in a definable such group K. Let $Q := N_R(K)$. Then Q is definable, and $S \leq Q$. Now S/K is a soluble subgroup of Q/K of derived length less than d, so there is a definable soluble group $\bar{H}$ with $S/K \leq \bar{H} \leq Q/K$. Let H be the preimage of $\bar{H}$ under the natural map $Q \to Q/K$. Then H is a definable soluble subgroup of Q containing S.

(ii) This is a straightforward adaptation of (i). □

Remark. In Proposition 3.2, the group H, obtained as in the proof, has derived series which has the same number of infinite factors as in the derived series of S. We do not know if in (i) one can require that S and H have the same derived length.

Proposition 3.3 *Let $G \in \mathcal{S}$. Then G has a largest soluble normal subgroup $R(G)$, and $R(G)$ is definable.*

Proof. First note that definability follows from the first assertion by Proposition 3.2 (ii).

To see the first assertion, let S be a definable soluble normal subgroup of G of maximal rank, and for $X \leq G$ with $S \leq X$ write $\bar{X}$ for X/S. Then if Q is any soluble normal subgroup of G, then QS/S is finite: indeed, we may suppose by Proposition 3.2 that Q is definable, so QS is definable, and the result then follows by maximality of the rank of S. Now work in $\bar{G}$. The union $\bar{N}$ of the finite normal subgroups of $\bar{G}$ is a definable BFC group, as $\exists^\infty$ is definable. By Theorem 2.9, $\bar{N}$ has a nilpotent class 2 definable subgroup $\bar{M}$ of finite index. Let M be the preimage of $\bar{M}$ in G. Then M is definable, soluble, and contains

S. So by maximality of rank it is a finite extension of S. Hence $\bar{M}$ is finite, and so is $\bar{N}$. Now let $\bar{R}$ be the product of the finite soluble normal subgroups of $\bar{G}$. Then $\bar{R}$ is a subgroup of $\bar{N}$ and so is finite and soluble, and its preimage $R = R(G)$ is the largest soluble normal subgroup of G, and is definable. □

Question. Is it true that if $G \in \mathcal{S}$ then G has a unique largest nilpotent normal subgroup N and that such N is definable?

The following result, combined with Proposition 3.3, gives some structure theory for members of $\mathcal{S}$ and $\mathcal{F}$. It builds on the results of Section 5 of [20], where groups with no finite conjugacy classes are considered.

Proposition 3.4 *Let $G \in \mathcal{S}$. Suppose $R(G) = 1$. Then*

(i) G has just finitely many finite conjugacy classes.

(ii) Any infinite normal subgroup of G contains an infinite definable normal subgroup of G.

(iii) G has just finitely many minimal normal subgroups, they are all definable, and $\mathrm{Soc}(G)$ *is definable.*

(iv) Any minimal normal subgroup of G is the direct product of finitely many isomorphic definable simple groups.

(v) If $G \in \mathcal{F}$, then the simple groups in (iv) are all finite, or Chevalley groups (possibly twisted) over a pseudofinite field.

Proof. (i) As $\exists^{\infty}$ is eliminated, the finite conjugacy classes of G have bounded size. If M is the union of the finite conjugacy classes, then M is a BFC group and so, by Theorem 2.9(ii), has a characteristic soluble subgroup of finite index. If this were infinite, this would contradict the assumption that $R(G) = 1$.

(ii) Let N be an infinite normal subgroup of G. By (i), N contains an infinite conjugacy class C of G. By Theorem 2.2, there is a definable normal subgroup K of G such that $K \leq \langle C \rangle$ and C/K is finite. In particular, K is infinite.

(iii) Definability of the minimal normal subgroups follows from (ii). By (i), there can be just finitely many finite minimal normal subgroups. By finiteness of the rank of G, Lemma 2.1, and the definability of minimal normal subgroups, G also has just finitely many infinite minimal normal subgroups. The definability of $\mathrm{Soc}(G)$ follows immediately.

(iv) Let N be a minimal normal subgroup of G. We may assume that N is infinite, since any minimal normal subgroup is characteristically simple (i.e. has no proper non-trivial characteristic subgroups); and it is well-known that any finite characteristically simple group is a direct product of (isomorphic) simple groups, which, being finite, are definable. Also, by (ii), N is definable.

We first show that N has no finite N-conjugacy classes. For suppose not. Then the union of the finite N-conjugacy classes of N is a characteristic subgroup of N, so normal in G, so, by minimality of N, equals N. Thus, as finite conjugacy classes have bounded size (by definability of $\exists^{\infty}$), N is a BFC group. Thus $R(N)$ is infinite, by Theorem 2.9. Since $R(N)$ is characteristic in N, it is normal in G, contradicting that $R(G) = 1$.

By compactness and Theorem 2.2, and the assumption that N is minimal normal, there is some $r > 0$ such that for every infinite conjugacy class C of N, there are $\varepsilon_1, \ldots, \varepsilon_s \in \{1, -1\}$ for some $s \leq r$ such that $C^{\varepsilon_1} \ldots C^{\varepsilon_s}$ is a subgroup of N. It follows that, if M is a normal subgroup of N, and C is an infinite conjugacy class of N contained in M, then M contains a definable normal subgroup of N of the form $C^{\varepsilon_1} \ldots C^{\varepsilon_s}$. Since such normal subgroups are uniformly definable, and $\mathrm{Th}(G)$ does not have the strict order property (as it is supersimple), there are no infinite chains of such normal subgroups. It follows that N has a minimal normal subgroup, Q say. By (ii), the group Q will be definable, and it is infinite by the last paragraph. Since distinct conjugates of Q are disjoint (by minimality of Q), it follows by finiteness of rank that $N = Q^{g_1} \times \ldots \times Q^{g_t}$ for some $t \in \mathbb{N}$ and $g_1, \ldots, g_t \in G$. Finally, Q is simple: for if X is a proper non-trivial normal subgroup of Q^{g_1} say, then as all the Q^{g_i} normalise X, X is normal in N, contrary to the minimality of Q.

(v) This is immediate from Proposition 2.14. □

Note that (ii) of Proposition 3.4 would be false without the assumption that $R(G) = 1$. Extraspecial groups (see Remark 2.15) provide a counterexample.

In general, the socle of a group in $\mathcal{S}$ may not be definable. An example is the group $\Sigma_{i\in\omega} C_{p_i} \oplus (\mathbb{Q}, +)$, which has socle $\Sigma_{i\in\omega} C_{p_i}$ and is an infinite model of the theory of the set of groups $\{\Sigma_{i=1}^{n} C_{p_i} : n \in \omega\}$; here $(p_i)_{i\in\omega}$ lists the primes. However, if $G \in \mathcal{S}$ has no finite conjugacy classes, then $\mathrm{Soc}(G)$ is definable, by [20, Lemma 5.3(2)]. We also have the following strengthening of [20, Lemma 5.2].

Lemma 3.5 *Let $G \in \mathcal{S}$, and let A be a minimal normal subgroup of G. Then A is definable.*

Proof. We may suppose that A is infinite. Let B be the union of the finite conjugacy classes of G, a definable normal subgroup of G. Either $A \leq B$, or $A \cap B = \{1\}$.

Suppose first that $A \leq B$. It follows easily from Theorem 2.9 that A is abelian, and so we write A additively. For any $n > 0$, nA is characteristic in A so normal in G, and it follows that A is elementary abelian or torsion-free. In fact, A is torsion-free, since otherwise, as sets a^G ($a \in A$) are finite, A would be finite. Now for each $n > 0$, as nA is normal in G, $nA = A$, so A is divisible. Thus, A is a vector space over $\mathbb{Q}$. Let $u_1 \in A \setminus \{0\}$. Since u_1^G is finite and generates A, the vector space has finite dimension. In particular, u_1^G contains a $\mathbb{Q}$-basis $\{u_1, \ldots, u_n\}$ of A, and every element v of u_1^G has the form $v = \Sigma_{i=1}^{n} \alpha_i u_i$, where $\alpha_i \in \mathbb{Q}$. Let r be the least common multiple of the denominators of all the elements of form α_1 as v ranges through u_1^G, and put $u := 2ru_1$. Then u^G generates a normal subgroup of G which is properly contained in A, contradicting minimality of A.

Thus, we may suppose $A \cap B = \{1\}$. Let $a \in A \setminus \{1\}$. Then by Theorem 2.2, there is a definable normal subgroup N of G with $N \leq \langle a^G \rangle$ and a^G/N finite. As a^G is infinite, $N \neq 1$, so by minimality $A = N$, and A is definable. □

4 Rank two Groups

In this section we investigate rank 2 groups in $\mathcal{M}$ and $\mathcal{F}$. Since the results all depend on Theorem 2.11 which uses unimodularity, our methods shed no light on rank 2 groups in $\mathcal{S} \setminus \mathcal{M}$. Our results extend those of [20], where it is shown that any rank 2 infinite ultraproduct of an asymptotic class is soluble-by-finite. Our eventual purpose, not yet achieved, is to obtain a positive answer to the following question.

Question 4.1 *Is every rank 2 group $G \in \mathcal{M}$ soluble-by-finite?*

First, note the following result [20, Lemma 4.6]. In [20] the theory is assumed to be an $S1$-theory (which includes an assumption that rank is definable), but definability of rank plays no role.

Proposition 4.2 *Let $G \in \mathcal{M}$ have rank 2. Then either G has a definable soluble subgroup of finite index and derived length at most 4, or G has a definable finite-by-simple subgroup of finite index, so interprets a simple rank 2 group in $\mathcal{M}$.*

In view of Proposition 4.2, we assume in the rest of this section that G is a simple group in $\mathcal{M}$ of rank 2. We obtain partial results, and derive a contradiction from the additional assumption of pseudofiniteness.

Lemma 4.3 *Suppose $G \in \mathcal{M}$ is simple of rank 2. Then G has no finite conjugacy classes.*

Proof. This is immediate from the simplicity of G and Corollary 2.10. □

Proposition 4.4 *Let $G \in \mathcal{M}$ be a simple group of rank 2. Suppose that Ω is a definable rank 1 set upon which G acts definably and transitively.*

(i) For each $x \in \Omega$, G_x acts on $\Omega \setminus \{x\}$ with finitely many infinite orbits and finitely many finite ones.

(ii) There is a unique maximally coarse definable G-congruence on Ω. It has blocks of finite size, and is given by $x \sim y :\Leftrightarrow [G_x : G_x \cap G_y] < \infty$ for all $x, y \in \Omega$. It is maximal among all proper G-congruences, and any G-congruence with finite blocks refines it.

Proof. Claim 1. Let $\approx$ be any definable G-congruence on Ω. If $\approx$ is not the universal congruence on Ω then $\approx$ has finite blocks.

Proof of Claim. Suppose for a contradiction that there is an infinite $\approx$-class. Then by transitivity of the action of G, all the $\approx$-classes must be infinite. Thus as $\approx$ is definable, and Ω has rank 1, there can be only finitely many classes. So we have found a finite set (the set of $\approx$-classes), on which G acts transitively. Then the kernel of this action is a nontrivial normal subgroup of G, so by simplicity of G there is in fact only one $\approx$-class, yielding the claim.

Consider now the relation $x \sim y :\Leftrightarrow [G_x : G_x \cap G_y] < \infty$. By Lemma 2.6, this is a definable G-congruence. Denote the $\sim$-class of $x \in \Omega$ by $\tilde{x}$.

By the orbit-stabilizer theorem

$$|\mathrm{Orb}_{G_x}(y)| < \infty \Leftrightarrow [G_x : G_x \cap G_y] < \infty \Leftrightarrow x \sim y$$

Claim 2. Each $\sim$-class is finite.

Proof of Claim. Suppose not. Then by Claim 1, Ω is a single $\sim$-class. It follows, as $\exists^\infty$ is definable, that the groups G_x^g (for $x \in \Omega$) are uniformly commensurable, so by Theorem 2.5 there is definable $N \lhd G$ commensurable with them. As $\mathrm{rk}(G_x) = 1$, $\mathrm{rk}(N) = 1$, contradicting simplicity of G.

To see (i), note that, by the claims, $\tilde{x}$ is finite, so there are finitely many finite G_x-orbits. Since Ω has rank 1, there can only be finitely many infinite orbits, yielding (i).

For (ii), let $\approx$ be any definable non-universal G-congruence on Ω. Then by Claim 1, the $\approx$-classes are finite, so if $x \approx y$ then $|G_x : G_{xy}| < \infty$, so $x \sim y$. Thus, the action of G on $\Omega/\sim$ is definably primitive. Hence, by Lemma 2.7, as $\mathrm{rk}(G) > \mathrm{rk}(\Omega/\sim)$, this action is primitive, so $\sim$ is a maximal proper G-congruence (among not necessarily definable ones). Finally, if $\approx$ is *any* G-congruence with finite blocks, then again if $x \approx y$ then $|G_x : G_{xy}| < \infty$, so $x \sim y$. □

Corollary 4.5 *Let $G \in \mathcal{M}$ be a simple rank 2 group, and C be a rank 1 subgroup of G. Then there is a maximal subgroup N of G which contains C, has rank 1, and is definable. Furthermore, every definable rank 1 subgroup of G containing C is contained in N.*

Proof. Let $C \leq G$ be of rank 1. Then the left coset space $X := \mathrm{Cos}(G : C)$ has rank 1, and G acts on it by left multiplication. Under this action, the stabiliser G_C is C. So let $\sim$ be the unique maximal definable G-congruence on $\mathrm{Cos}(G : C)$ defined in Proposition 4.4, let $[aC]$ denote the $\sim$-class of $aC \in X$, and let N be the stabilizer of $[C]$ under the action of G on $X/\sim$. Clearly $C \leq N$. Notice that G acts definably primitively on $X/\sim$, so there is no definable H with $N < H < G$. Hence, by Lemma 2.7, N is maximal in G. Since $X/\sim$ is infinite, $\mathrm{rk}(N) < 2$, so $\mathrm{rk}(N) = 1$.

For the second assertion, suppose that also $C \leq H$ where H is some rank 1 subgroup of G. Now H acts transitively by conjugation on $\{hCh^{-1} : h \in H\}$, and since C is of finite index in H, the orbit-stabilizer theorem shows that this family is finite. Therefore the kernel of this action is infinite and is contained in every hCh^{-1}. Thus H-conjugates of C have pairwise-infinite intersection, so are commensurable. Since hCh^{-1} is the stabiliser of the element hC of X (in the G-action on X), it follows that for any $h_1, h_2 \in H$ we have $h_1C \sim h_2C$, and $\mathrm{Cos}(H : C) \subseteq [C]$. Thus H (in the action on $X/\sim$) fixes $[C]$, so $H \leq N$.□

Lemma 4.6 *Let $G \in \mathcal{M}$ be a simple rank 2 group, and N be a rank 1 maximal definable subgroup of G. Then*

(i) $\mathrm{Conj}(N) := \{aNa^{-1} : a \in G\}$ *has rank 1.*

(ii) Any two non-identical conjugates aNa^{-1} and bNb^{-1} have finite intersection.

(iii) Each $x \in G \backslash \{1\}$ appears in only finitely many conjugates of N.
(iv) N has finite intersection with each rank 1 conjugacy class.

Proof. (i) G acts on $\mathrm{Conj}(N)$ by conjugation. Now $aNa^{-1} = N \Leftrightarrow a \in N$, since N is maximal and has rank 1, but is not normal, and so is self-normalising. This shows that there is a definable bijection between the coset space $\mathrm{Cos}(G : N)$ and $\mathrm{Conj}(N)$. As $\mathrm{rk}(N) = 1$, also $\mathrm{Cos}(G : N)$ has rank 1, and hence so does $\mathrm{Conj}(N)$.

(ii) For any $a \in G$, aNa^{-1} is maximal in G and has rank 1. But if $H = bNb^{-1} \cap aNa^{-1}$ is infinite, then by Corollary 4.5 H lies in a unique definable rank 1 maximal subgroup of G. So $H = bNb^{-1} = aNa^{-1}$.

(iii) Suppose this is false. Let

$$S := \{x \in G \setminus \{1\} : \{N^g : g \in G, x \in N^g\} \text{ is infinite}\}.$$

Then S is a union of conjugacy classes, and is non-empty, so as G has no finite conjugacy classes, S is infinite (and definable).

Consider the definable set $Y \subseteq \mathrm{Conj}(N) \times \mathrm{Conj}(N) \times G$ where

$$Y = \{(xNx^{-1}, yNy^{-1}, z) : x, y \in G, z \in xNx^{-1} \cap yNy^{-1}\}$$

We compute $\mathrm{rk}(Y)$. By considering the first two coordinates first, and using (ii), $\mathrm{rk}(Y) = 2$. On the other hand, by considering the third coordinate first, and choosing $z \in S$, we have $\mathrm{rk}(Y) = 3$. This contradiction proves (iii).

(iv) Suppose that N has infinite intersection with some rank 1 conjugacy class C. This holds also for each conjugate of N. Since there are infinitely many conjugates of N, and any two have finite intersection, this contradicts that C has rank 1 (working e.g. with the definition of S1 rank). □

Definition 4.7 In G, we call elements with conjugacy class of rank at most 1 *good*, and those with rank 2 conjugacy class *bad*.

Note that bad elements have centraliser of finite bounded size, so the set of bad elements is definable, so 'good' is also definable.

Lemma 4.8 *Let $G \in \mathcal{M}$ be simple of rank 2. If $x \in G$ is good then x lies in a unique maximal rank 1 subgroup of G.*

Proof. Since x is good, $\mathrm{rk}(C_G(x)) = 1$. So let N be the unique maximal definable rank 1 group containing $C_G(x)$ (see Corollary 4.5). Now suppose M is another maximal rank 1 subgroup containing x. Consider the following set of conjugates of M:

$$\mathcal{C} = \{aMa^{-1} : \ a \in C_G(x)\}$$

Now $\mathcal{C}$ is a collection of maximal rank one subgroups each of which contains the element x. But by Lemma 4.6 (iii) $\mathcal{C}$ must be finite. Let J be the kernel of the action by conjugation of $C_G(x)$ on $\mathcal{C}$. Then J is a subgroup of finite index in $C_G(x)$. Also,

$$\forall j \in J \ \ jMj^{-1} = M$$

Since M is self-normalising, $J \subseteq M$. But J has rank 1 so lies in a unique maximal rank 1 subgroup of G. Since $J \subseteq N$ we deduce $N = M$. □

Lemma 4.9 *Let $G \in \mathcal{M}$ be simple of rank 2. Suppose N is a maximal definable rank one subgroup of G. Then the set of good elements which lie in N is a definable normal subgroup of N of finite index.*

Proof. By Theorem 2.11, the union of the finite N-conjugacy classes in N is an infinite definable characteristic subgroup of N. Thus, it suffices to show that if $x \in N$, then x is good if and only if x^N is finite. For this, note that $\mathrm{rk}(\mathrm{Conj}(N)) = 1$ and

$$x^G = \bigcup_{aNa^{-1} \in \mathrm{Conj}(N)} ax^N a^{-1}$$

Now if $x \in N$ is good, then $C_G(x) \leq N$ by Lemma 4.8, and it follows that $\mathrm{rk}(C_G(x)) = \mathrm{rk}(N) = 1$ and hence x^N is finite. On the other hand, if x^N is finite, then as $\mathrm{rk}(\mathrm{Conj}(N)) = 1$ we clearly have $\mathrm{rk}(x^G) = 1$. □

By Lemma 4.9 and Corollary 4.5, if $G \in \mathcal{M}$ is simple of rank 2 and M is any rank 1 definable subgroup, then the set of all good elements of M forms a definable subgroup of M of finite index. We denote the latter by M^o.

Lemma 4.10 *Let $G \in \mathcal{M}$ be simple of rank 2. Then G contains infinitely many rank 1 conjugacy classes.*

Proof. First, by Proposition 2.12, there is non-identity $x \in G$ with $C_G(x)$ infinite, and so $\mathrm{rk}(x^G) \leq 1$. Thus, by Lemma 4.3, G has a rank 1 conjugacy class, namely x^G. Since $\mathrm{rk}(C_G(x)) = 1$, G has a rank 1 maximal subgroup, N say.

Suppose that G has just finitely many conjugacy classes of good elements. Then as N is infinite, N^o has infinite intersection with some conjugacy class C of good elements. This contradicts Lemma 4.6(iv).
□

Lemma 4.11 *Let $G \in \mathcal{M}$ be simple of rank 2. Then the collection of maximal rank 1 subgroups divides up into finitely many disjoint families of the form* $\mathrm{Conj}(N)$.

Proof. Suppose $\{N_i : i \in I\}$ is a set of maximal definable rank 1 subgroups, none of which is conjugate to any of the others. We claim that the N_i are *uniformly* definable, i.e. that there are finitely many formulas $\varphi_i(x, \bar{y}_i)$ such that each N_i has the form $\varphi_j(G, \bar{a}_j)$ for some $\bar{a}_j \in G^{l(\bar{y}_j)}$. To see this, observe that as the set of good elements is definable, the family S of rank 1 groups $\{C_G(x) : x \text{ good}\}$ is uniformly definable. For every maximal rank one group N, there is $C \in S$ with $C \leq N$ (see the proof of Lemma 4.9). The actions of G on the coset spaces $\mathrm{Cos}(G : C)$ are uniformly definable. Since $\exists^\infty$ is definable, the maximal equivalence relation $\sim$ on $\mathrm{Cos}(G : C)$ with finite classes is also uniformly definable (as C varies). So N is uniformly defined as the stabiliser of the $\sim$-class of C in the action of G on $\mathrm{Cos}(G : C)$; see Corollary 4.5.

Now $\{\mathrm{Conj}(N_i) : i \in I\}$ is a collection of disjoint families of conjugate subgroups. For any $i \in I$, $\mathrm{rk}(\bigcup_{a \in G} aN_i^o a^{-1}) = 2$ by Lemma 4.6(i), (ii). Also for any $i, j \in I$ such that $i \neq j$ we have

$$\Big(\bigcup_{a\in G} aN_i^o a^{-1}\Big) \cap \Big(\bigcup_{a\in G} aN_j^o a^{-1}\Big) = \{1\}$$

by Lemma 4.8. Since $\mathrm{rk}(G) = 2$, it follows by rank considerations and the uniform definability of the N_i that $\mid I \mid < \infty$. $\square$

Lemma 4.12 *Let $G \in \mathcal{M}$ be simple of rank 2. Then there is a definable rank 1 subgroup H of G which consists solely of good elements, such that in the action of G by left multiplication on the coset space* $\mathrm{Cos}(G : H)$*, all the bad elements of G act fixed-point-freely.*

Proof. There is at least one good element by 4.10, so there is at least one maximal definable rank 1 group N, and we may put $H := N^o$.

Suppose now that $a \in G$ and $abH = bH$. So $b^{-1}ab \in H$ and $a \in bHb^{-1}$. But then a is conjugate to a good element, so is good. $\square$

Proposition 4.13 *Let $G \in \mathcal{M}$ be simple of rank 2. Then G contains at least one rank 2 conjugacy class.*

Proof. Suppose not. Then all non-identity conjugacy classes have rank 1. We may suppose that G is ω-saturated. For elementary extensions will preserve simplicity of G by Proposition 4.2, and existence of rank 2 conjugacy classes is equivalent to existence of elements with finite centraliser, which is first order expressible.

Let $x \in G\backslash\{1\}$. Then as G is simple, x^G generates the whole of G. Moreover, by compactness and ω-saturation, it does so in finitely many steps. Therefore for some minimal $n \geq 1$ there are $\varepsilon_1 \ldots, \varepsilon_{n+1} \in \{-1, 1\}$ and $Y := \prod_{i=1}^n (x^{\varepsilon_i})^G$ such that $\mathrm{rk}(Y) = 1$ and $\mathrm{rk}(Y \cdot (x^{\varepsilon_{n+1}})^G) = 2$. Without loss of generality (by exchanging x with x^{-1} if necessary) we may assume that $\varepsilon_{n+1} = 1$.

Notice that Y is closed under conjugation, and is therefore a disjoint union of finitely many rank one conjugacy classes. Similarly $Y \cdot x^G$ is a union of conjugacy classes, say $Y \cdot x^G = \bigcup\{D_i : i \in I\}$. For each $i \in I$ define $E_i := \{y \in Y : yx \in D_i\}$. Note that if $y \in E_i$ and $b \in C_G(x)$ then $y^b x = y^b x^b = (yx)^b \in D_i$, so $y^{C_G(x)} \subseteq E_i$.

Now each E_i is non-empty. For suppose $y' \cdot gxg^{-1} \in D_i$, where $y' \in Y$. Then conjugating by g we find that $g^{-1}y'g \cdot x \in D_i$ and therefore $g^{-1}y'g \in E_i$.

We claim that for all but finitely many $y \in Y$ we have: if $y \in E_i$ then E_i is infinite. Indeed, suppose that $U \subseteq Y$ is a conjugacy class. Then for only finitely many $y \in U$ may we have $\mathrm{rk}(C_G(x) \cap C_G(y)) = 1$, by Proposition 4.4 applied to the action by conjugation of G on U. Thus for cofinitely many $y \in U$ we have that $C_G(x) \cap C_G(y)$ is finite, and for all such $y \in U$, if $y \in E_i$ then $y^{C_G(x)} \subseteq E_i$ (as noted above), so E_i is infinite. Recalling that there are only finitely many such $U \subseteq Y$ yields the claim.

Hence as Y has rank 1 and the E_i partition Y, it follows that I must be finite. But then as $Y \cdot x^G$ has rank 2, it follows that at least one of the conjugacy classes D_i must have rank 2, a contradicting our assumption. □

The following result, a strengthening of the last proposition but under a stronger hypothesis, is not used elsewhere in the paper. It seems to be the only place in this section where we need measurability, rather than just unimodularity or pseudofiniteness.

Proposition 4.14 *Assume that $G \in \mathcal{M}$ is measurable simple of rank 2. Let N be a rank one maximal subgroup of G. Then N contains a bad element.*

Proof. Suppose for a contradiction that all the elements of N are good. We shall write $\mu(X)$ for the measure (in the sense of measurable structures, as in [34]) of a definable set X. We may normalise to ensure $\mu(G) = 1$.

We show that under these assumptions, the conjugates of N cover G, contradicting the last result. Consider the definable map

$$\varphi : \ \mathrm{Cos}(G : N) \mapsto \mathrm{Conj}(N) \quad \text{ where } aN \mapsto aNa^{-1}$$

Since N is self-normalising, φ is bijective. It follows that $\mu(\mathrm{Cos}(G : N)) = \mu(\mathrm{Conj}(N))$.

Now let $X := \bigcup_{a \in G} aNa^{-1}$. Then

$$X \backslash \{1\} = \bigcup_{a \in G} (aNa^{-1} \backslash \{1\})$$

and by 4.8, as all elements of N are good, the sets on the right are equal or disjoint. So by elementary properties of rank and measure we have $\dim(X) = 2$ and

$$\begin{aligned} \mu(X) = \mu(X \backslash \{1\}) = \quad & \mu(N \backslash \{1\}) \cdot \mu(\mathrm{Conj}(N)) \\ & = \mu(N) \cdot \mu(\mathrm{Conj}(N)) \\ & = \mu(N) \cdot \mu(\mathrm{Cos}(G : N)) \\ & = 1 \end{aligned}$$

as $\mu(G) = 1$ by our assumption.

Suppose now that $y \in G\backslash X$. Then y cannot be bad, as then y^G would be a rank two set disjoint from X which is impossible: indeed, we would have $\mu(y^G) > 0$, $\mu(X) = \mu(G)$, and $\mu(y^G)+\mu(X) \le \mu(G)$. This shows that all points of G are good, which is impossible by 4.13 above. □

We now answer Question 4.1 assuming $G \in \mathcal{F}$. We do not use CFSG.

Lemma 4.15 *Let $G \in \mathcal{F}$ be simple of rank 2. Then any two maximal rank 1 definable subgroups of G are conjugate.*

Proof. Suppose that G has non-conjugate maximal rank 1 subgroups N_1 and N_2. By Lemma 2.7, N_1 and N_2 are abstractly maximal, and also are uniformly maximal. Suppose that $\varphi_i(x, \bar{a}_i)$ defines N_i for $i = 1, 2$. It follows that there is a sentence σ in $\mathrm{Th}(G)$ which expresses that there are subgroups $\varphi_i(G, \bar{a}_i)$ (for $i = 1, 2$) which are maximal and non-conjugate. We may suppose that G is a nonprincipal ultraproduct $\Pi_{j\in J} G_j/\mathcal{U}$.

Let $C_i := N_i^o$, for $i = 1, 2$. Then $N_i = N_G(C_i)$. Let $x \in C_1\backslash\{1\}$. Then as $C_G(x)$ has rank 1 and x is good so lies in at most one maximal rank 1 group (Lemma 4.8), it follows that $C_G(x) \le N_1$, and moreover $|C_1 : C_1 \cap C_G(x)|$ is finite.

Let G act on x^G by conjugation, and let $X := x^G/\sim$, where $\sim$ is the fundamental congruence: $x_1 \sim x_2 :\Leftrightarrow C_G(x_1) \cap C_G(x_2)$ is infinite. We examine the action of the C_i on X.

Claim 1. Every orbit of C_2 on X is regular.

Proof of Claim. Suppose $h \in G$, and $g \in C_2$, and $g(hxh^{-1})g^{-1} \sim hxh^{-1}$, that is, conjugation by g fixes $x^h/\sim$. Then $h^{-1}ghxh^{-1}g^{-1}h \sim x$, so $C_G(x^{h^{-1}gh}) \cap C_G(x)$ is infinite. Thus $C_G(x^{h^{-1}gh}) \cap C_1$, and $(C_G(x))^{h^{-1}gh} \cap C_1$ and so also $C_1^{h^{-1}gh} \cap C_1$ are all infinite. It follows that $N_1^{h^{-1}gh} = N_1$, as otherwise $C_1^{h^{-1}gh} \cap C_1$ is a rank 1 subgroup of two distinct maximal rank one subgroups, contrary to Corollary 4.5. Thus, $h^{-1}gh \in N_1$, so $g \in hN_1h^{-1}$. Since N_1 and N_2 are non-conjugate, it follows that $g = 1$.

Claim 2. There is one orbit of C_1 on X of size 1, namely the $\sim$-class of x, and the rest are regular.

Proof of Claim. First note that if $g \in C_1$ then $C_G(x) \cap C_G(x^g)$ is infinite, so $x \sim x^g$; that is, $x/\sim$ is a singleton orbit of C_1.

Arguing as in Claim 1, if $h \in G$, and $g \in C_1$ with $g(hxh^{-1})g^{-1} \sim hxh^{-1}$, then $g \in N_1^h \cap N_1$. Now there are 2 cases:

(i) $hN_1h^{-1} = N_1$, and as N_1 is self-normalising, $h \in N_1$, in which case $hC_1h^{-1} = C_1$, and thus, as $C_G(x)$ and $hC_G(x)h^{-1}$ are both commensurable with C_1, we have that $hC_G(x)h^{-1} \cap C_G(x)$ is infinite, and so $hxh^{-1} \sim x$.

(ii) $hN_1h^{-1} \ne N_1$, in which case $g = 1$ by Lemma 4.8.

By Los's Theorem, for ultrafilter-many $j \in J$ the sentence σ holds, with respect to the groups $N_i^{(j)} := \varphi_i(G^{(j)}, \bar{a}_i^{(j)})$ (for $i = 1, 2$, and for some parameters $\bar{a}_i^{(j)}$ from $G^{(j)}$). We drop the superscript j, so work with a finite group G, and non-conjugate maximal subgroups N_1, N_2. Let C_i be as above (i.e. defined by the same formula defining it in the ultraproduct), with $N_i = N_G(C_i)$, and assume $j \in J$ is chosen that Claims 1 and 2 now hold in this finite situation, where $x \in C_1 \setminus \{1\}$ and $X := x^G$.

Putting the claims together, there are fixed (on an ultrafilter set) strictly positive integers a_1 and b_1, where $|X| = a_1|C_2| = b_1|C_1| + 1$. Similarly there are a_2 and b_2 where $a_2|C_1| = b_2|C_2| + 1$. Hence $a_1a_2|C_2| = b_1a_2|C_1| + a_2$, and $b_1a_2|C_1| = b_1b_2|C_2| + b_1$, and so $a_1a_2|C_2| = b_1b_2|C_2| + b_1 + a_2$, and so $|C_2| = \frac{b_1+a_2}{a_1a_2-b_1b_2}$ (note that $a_1a_2 \neq b_1b_2$, as otherwise $b_1 + a_2 = 0$, contradicting that they are strictly positive). But this is a contradiction, as a_1, a_2, b_1, b_2 are fixed integers, and $|C_2|$ can be made arbitrarily large. $\Box$

The following theorem completes the proof of Theorem 1.2.

Theorem 4.16 *Let $G \in \mathcal{F}$ have rank 2. Then G has a definable soluble subgroup of finite index.*

Proof. We may suppose that G is an ultraproduct $\Pi_{j\in J}G_j/\mathcal{U}$ of finite groups. In addition we may suppose by Proposition 4.2 that G is simple.

By the analysis in this section, we may also suppose that G has finitely many (definable) rank 2 conjugacy classes, and infinitely many rank 1 classes. Also, by Lemma 4.15, all maximal rank 1 definable subgroups are conjugate. In particular, maximal rank 1 definable subgroups of G are uniformly definable, say by the formulas $\varphi(x, \bar{y})$.

By Lemma 2.7, if C is a maximal rank 1 subgroup of G, then C is uniformly maximal. That is, there is $t > 0$ such that if $g, h \in G \setminus C$, then $h = c_1g^{\pm1}c_2g^{\pm1}\dots c_t$, where $c_1, \dots, c_t \in C$. Thus, there is a formula $\psi_\varphi(\bar{y})$ expressing that the group defined by $\varphi(x, \bar{y})$ is maximal, via uniform maximality with parameter t. In particular, for any group H and $\bar{a}$ from H, if $H \models \psi_\varphi(\bar{a})$ then $\varphi(H, \bar{a})$ is a maximal subgroup of H. Let $\varphi^*(x, \bar{y})$ be the formula $\varphi(x, \bar{y}) \wedge \psi_\varphi(\bar{y})$.

There is n such that if $x \in G \setminus \{1\}$ and $|C_G(x)| > n$, then $C_G(x)$ is infinite. Since G is simple, it follows that for such x, $C_G(x)$ has rank 1, so is contained in a maximal rank 1 definable subgroup of G. There is a sentence σ which expresses this: namely:

$$\forall x((x \neq 1 \wedge |C_G(x)| > n) \to \exists \bar{z} \forall y([x, y] = 1 \to \varphi^*(y, \bar{z}))).$$

We may suppose that σ holds in all G_j. We may also suppose that for any G_j and tuples $\bar{a}$, $\bar{b}$ from G_j such that $G_j \models \psi_\varphi(\bar{a}) \wedge \psi_\varphi(\bar{b})$, the maximal subgroups $\varphi(G_j, \bar{a})$ and $\varphi(G_j, \bar{b})$ are conjugate in G_j, since the corresponding assertion is true of G and is first order expressible.

Let P be a Sylow p-subgroup of some G_j. Then there is $x \in Z(P) \setminus \{1\}$. Thus, if $|P| > n$, then as $P \leq C_{G_j}(x)$ and $G_j \models \sigma$, there is some $\bar{a}$ in G_j such that $P \subseteq \varphi^*(G_j, \bar{a})$. Let C_j be the subgroup of G_j defined by $\varphi^*(x, \bar{a})$. Then $|P|$ divides $|C_j|$. Suppose $|G_j| = p_1^{a_1} \dots p_r^{a_r}$, the prime power decomposition. Then, by the conjugacy of φ^*-definable subgroups, for all i such that $p_i^{a_i} > n$, it follows that $p_i^{a_i}$ divides $|C_j|$. Hence, $|G_j : C_j| \leq n^n$. Since this holds for all i, it follows that $|G : C| \leq n^n$. This contradicts that $\mathrm{rk}(C) = 1$. □

We have not managed to prove that there is no simple rank 2 group $G \in \mathcal{M}$. The proof of Theorem 2.11 suggests that that there might be one argument to handle the case when G has an involution, and another to handle the involution-free case. We conclude this section with some remarks on the first case. The goal has been to eliminate involutions using arguments developed in [4, 16], but this has not been achieved.

First we recall an easy fact:

Fact 4.17 *Let G be a group. Then every $x \in G$ lies in a definable abelian subgroup of G.*

Proof. It suffices to take the double-centralizer $C_G(C_G(x))$ of x. □

Lemma 4.18 *Let $G \in \mathcal{M}$. Suppose that $i, j \in G$ are involutions. Let $x = ij$, and suppose that $x \neq 1$. For any definable abelian group A such that $x \in A$, either there is an involution $k \in A$ such that x commutes with both i and j, or i and j are conjugate by an element of A.*

Proof. Let A be a definable abelian subgroup of G with $x \in A$. Let $y \in G$ be an involution and let $B_y = \{a \in A : \ yay = a^{-1}\}$. Notice this is a definable subgroup of A. Definability is clear. For closure under inversion, suppose $a \in B_y$. Then $ya^{-1}y \cdot yay = 1$ so $ya^{-1}y = (yay)^{-1} = (a^{-1})^{-1} = a$. To see closure under multiplication, suppose $a_1, a_2 \in B_y$. Then $ya_1a_2y = ya_1y \cdot ya_2y = a_1^{-1}a_2^{-1}$ and the latter equals $a_2^{-1}a_1^{-1}$ by commutativity of A.

Now notice that $ixi = i \cdot ij \cdot i = ji = x^{-1}$ and $jxj = j \cdot ij \cdot j = ji = x^{-1}$ and so $x \in B_i \cap B_j$, and in particular $B := B_i \cap B_j$ is a non-trivial definable subgroup of A.

If B contains an involution k, then $iki = k^{-1} = k$ and also $jkj = k$, so k commutes with both i and j. Otherwise B has no involutions. It follows by Lemma 2.13 that each element x of B has a unique square root y in B. Now we have $y^{-1}iy = iiy^{-1}iy = i(iy^{-1}i)y = iy^2 = ix = j$. □

Lemma 4.19 *Let $G \in \mathcal{M}$ be simple of rank 2. Then G does not have a rank 2 conjugacy class of involutions.*

Proof. Suppose such a conjugacy class g^G exists. We fix g and consider the set $X_g := \{gh : \ h \in g^G \ \wedge \ h$ does not commute with $g\}$. Then $\mathrm{rk}(X_g) = 2$, since $\mathrm{rk}(g^G) = 2$, and $C_G(g)$ is finite.

By Lemma 4.8, if x is good then there is a unique maximal definable rank 1 subgroup containing x, denoted by N_x.

Let $Y_g := \{gh \in X_g : gh$ is good$\}$, and $S_g := \{N_{gh} : gh \in Y_g\}$. Suppose for a contradiction that $\mathrm{rk}(Y_g) = 2$. Then $\mathrm{rk}(S_g) = 1$, by Lemmas 4.6(i) and

4.11. For each $gh \in Y_g$ we have $g \cdot gh \cdot g^{-1} = g \cdot gh \cdot g = hg = (gh)^{-1}$. Now $N_{gh} = N_{(gh)^{-1}}$ so it follows by Lemma 4.8 that $gN_{gh}g^{-1} = N_{gh}$. But N_{gh} is self-normalising and so $g \in N_{gh}$. Thus g lies in every element of S_g. Since, by Lemma 4.11 there are only finitely many distinct sets $\mathrm{Conj}(N)$ for maximal definable rank 1 groups N, it follows that S_g contains some group N and a rank one set of its conjugates. But then g lies in all these conjugate subgroups, which contradicts Lemma 4.6.

So now for $g_i \in g^G$ we may consider the set $S_2(g_i) = \{x : g_i x g_i = x^{-1} \wedge \mathrm{rk}(x^G) = 2\}$. Then from above $\mathrm{rk}(Y_{g_i}) = 1$, so $\mathrm{rk}(X_{g_i} \setminus Y_{g_i}) = 2$. For any $g_i h \in X_{g_i} \setminus Y_{g_i}$ we have that $\mathrm{rk}((g_i h)^G) = 2$, and that $g_i(g_i h)g_i = hg_i = (g_i h)^{-1}$, and so $g_i h \in S_2(g_i)$. So $\mathrm{rk}(S_2(g_i)) = 2$.

Let k be the maximal size of the centralizer of an element of a rank two conjugacy class. Suppose $x \in \bigcap_{i=1}^{k+2} S_2(g_i)$ for elements $g_1, \ldots, g_{k+2} \in g^G$. Then for each $i \leq k+1$ we have $g_i g_{k+2} x g_{k+2} g_i = x$, which contradicts the maximality of k. Thus $\{S_2(g_i) : g_i \in g^G\}$ is an infinite, $(k+2)$-inconsistent family of rank 2 definable sets, which contradicts the assumption that $\mathrm{rk}(G) = 2$. □

Proposition 4.20 *Let $G \in \mathcal{M}$ be simple of rank 2. If there are any involutions then there is exactly one conjugacy class of involutions, and this class has rank 1.*

Proof. Suppose there are two distinct conjugacy classes of involutions, g^G and h^G, say. We know from 4.18 that for every $x \in g^G$ and every $y \in h^G$, x and y both commute with a third involution, call it z_{xy}. Now by Lemma 4.19 z_{xy} must have rank 1 conjugacy class. So z_{xy} is an element of a unique maximal definable rank 1 group $N_{z_{xy}}$. Since $z_{xy} = x z_{xy} x^{-1} \in x N_{z_{xy}} x^{-1}$ it follows that $N_{z_{xy}} = x N_{z_{xy}} x^{-1}$. But then by the self-normalization of $N_{z_{xy}}$ it follows that $x \in N_{z_{xy}}$. So $N_x = N_{z_{xy}}$. Similarly, $N_y = N_{z_{xy}}$, so $N_x = N_y$. But this was for arbitrary $x \in g^G$ and $y \in h^G$. Fixing x and picking two distinct $y_1, y_2 \in h^G$, we may deduce that $N_{y_1} = N_{y_2} = N$, say. Thus N, and hence any conjugate of N, contains h^G, which contradicts Lemma 4.6(iv). □

The techniques developed in [4, 16] (the 'Borovik-Cartan decomposition' – see also [13, Section 7]) take the analysis further when there are involutions. It can be shown that if $G \in \mathcal{M}$ is simple of rank 2 and i is an involution of G, then each right coset of $C_G(i)$ (apart from finitely many) contains exactly one involution, of the form $i(i^g i)^{-\frac{1}{2}}$; each such coset also contains exactly one non-involutory element inverted by i, namely $(i^g i)^{-\frac{1}{2}}$ (a bad element of odd order). Analogous problems are also treated, slightly differently, in [38].

5 Groups acting on a rank one set

Our main goal in this section is to prove the following theorem (Theorem 1.3 of the Introduction).

Theorem 5.1 *Let $(X, G) \in \mathcal{F}$ be a definably primitive permutation group, and suppose that* $\mathrm{rk}(X) = 1$*. Let $S = \mathrm{Soc}(G)$. Then one of the following holds.*

(i) $\mathrm{rk}(G) = 1$*, and S is divisible torsion-free abelian or elementary abelian, has finite index in G, and acts regularly on X.*

(ii) $\mathrm{rk}(G) = 2$*. Here S is abelian so regular and identified with X. There is an interpretable pseudofinite field F with additive group X, and G is a subgroup of* $\mathrm{AGL}_1(F)$ *of finite index, with the natural action.*

(iii) $\mathrm{rk}(G) = 3$*. There is an interpretable pseudofinite field F, $S = \mathrm{PSL}_2(F)$,* $\mathrm{PSL}_2(F) \leq G \leq \mathrm{P\Gamma L}_2(F)$*, and G has the natural action on* $\mathrm{PG}_1(F)$*.*

Theorem 5.1 follows from Propositions 5.3, 5.11, and 5.12.

The theorem clearly has implications without the definable primitivity assumption. For example, just assuming transitivity, as $\mathrm{rk}(X) = 1$ any definable G-congruence has finite classes, or finitely many classes. Without definable primitivity, there is no bound on $\mathrm{rk}(G)$: for example, $\mathrm{PSL}_2(F)^n$ (F pseudofinite) has rank $3n$ and acts transitively on the disjoint union of n copies of $\mathrm{PG}_1(F)$, which has rank 1.

5.1 Preliminaries for Theorem 5.1

We begin with a standard observation.

Lemma 5.2 *Let $(X, G) \in \mathcal{S}$ be a definably primitive permutation group, and let A be a non-trivial definable abelian normal subgroup of G. Then*

(i) A acts regularly on X, and has no proper non-trivial definable characteristic subgroups.

(ii) A is an elementary abelian p-group for some prime p or a torsion-free divisible abelian group, so (written additively) may be viewed as a vector space over a field F (which is $\mathbb{F}_p$ or $\mathbb{Q}$).

(iii) If we identify A with X (by first identifying $0 \in A$ with some chosen $x \in X$) then we may identify (X, G) with (A, AG_0) where G_0 is the stabiliser of $0 \in A$. There are no definable proper non-trivial FG_0-submodules of A.

Proof. (i) As G is definably primitive, A is transitive on X (for the orbits of A are the classes of a definable G-congruence). Thus A is regular on X, as A is abelian. If A had a proper non-trivial definable characteristic subgroup B, then its orbits would be the blocks of a proper non-trivial G-congruence, contradicting definable primitivity.

(ii) This follows easily from (i), as A is abelian and 'definably characteristically simple': for any $n \in \mathbb{N}$, the group nA equals $\{0\}$ or A.

(iii) Again this is standard, and elementary. □

We first handle the (easy) case when $\mathrm{rk}(G) = 1$. The following proposition yields Theorem 5.1(i). Here, no pseudofiniteness assumption is needed.

Proposition 5.3 *Assume $(X, G) \in \mathcal{M}$ is a definably primitive permutation group with* $\mathrm{rk}(X) = \mathrm{rk}(G) = 1$*. Then G has a definable normal subgroup A of* *finite index which is divisible torsion-free abelian or elementary abelian and acts regularly on X. If G_0 denotes the stabiliser in G of $0 \in A$, and A is viewed as a vector space over a prime field, then $G_0 \leq \mathrm{GL}(A)$ is finite and irreducible, and $G = AG_0$, so that if X is identified with A as in Lemma 5.2, then G acts on A as an affine group (A by translation, G_0 by conjugation).*

Proof. By Theorem 2.9, G has a definable non-trivial abelian normal subgroup A. By Lemma 5.2, A acts regularly on X, so $\mathrm{rk}(A) = 1$ and $|G : A| < \infty$. The remaining assertions also follow from Lemma 5.2. □

Remark 5.4 We give an example interpretable in pseudofinite fields where $G \neq A$. Let K be a pseudo-finite field of characteristic 0, and denote by K^+ its additive group, and $K^\times$ its multiplicative group. Let $T \leq K^\times$ be exactly $T = \{\pm 1\}$. Let $G = K^+T$, where T acts on K^+ by conjugation, and view G as a subgroup of $\mathrm{AGL}_1(K)$ acting on K. Then G and K both have rank one. Here $|T| = 2$ but T is still definably maximal in G. To see the latter, note that if T were not definably maximal then G would be definably imprimitive, so there would be a proper non-trivial T-invariant definable subgroup of K^+. This would be infinite and of infinite index (as K is of characteristic 0 so K^+ is divisible) but the latter is impossible as $\mathrm{rk}(K^+) = 1$.

Next, we collect some general facts about the case when $\mathrm{rk}(G) > 1$. Note that in this case $\mathrm{rk}(G) > \mathrm{rk}(X)$, so by Lemma 2.7, definable primitivity of $(X, G) \in \mathcal{M}$ implies primitivity. This will be used without explicit mention.

Lemma 5.5 *Assume $(X, G) \in \mathcal{S}$ is a definably primitive permutation group with $\mathrm{rk}(G) > \mathrm{rk}(X) = 1$. Let $x \in X$. Then G_x has finitely many orbits on $X \setminus \{x\}$, all of which are infinite.*

Proof. Define $\sim$ on X, putting $x \sim y$ if and only if $|G_x : G_{xy}| < \infty$. By Lemma 2.6, $\sim$ is a definable G-congruence. If all $\sim$-classes are singletons, then each G_x-orbit on $X \setminus \{x\}$ is infinite, so as $\mathrm{rk}(X) = 1$ there are finitely many such orbits. So suppose for a contradiction that $\sim$ is non-trivial. By definable primitivity, $\sim$ is the universal congruence. Hence, as $\exists^\infty$ is definable, there is $n \in \mathbb{N}$ such that $|G_x : G_{xy}| < n$ for all distinct $x, y \in X$. It follows that the conjugates of G_x are commensurable, so by Schlichting's Theorem (Theorem 2.5) there is $N \triangleleft G$ commensurable with G_x. Since G_x is infinite, so is N, so since G is primitive on X (by Lemma 2.7), N is transitive. This contradicts that $|N : N \cap G_x|$ is finite and $N \cap G_x$ fixes x. (This argument is essentially in [20, Section 5].) □

Remark 5.6 In [31, Section 7] there is a description, which is close to a classification, of primitive ultraproducts of finite permutation groups. It is pointed out there that if (X, G) is an ultraproduct of finite primitive permutation groups and G has finitely many orbits on X^2, then (X, G) is primitive, so satisfies the description in [31]. In fact, suppose that (X, G) is *any* definably primitive permutation group in $\mathcal{F}$ with $\mathrm{rk}(G) > \mathrm{rk}(X)$. Then (X, G) has an ω-saturated elementary extension (X^*, G^*) with the same properties. By Lemma 2.7, (X^*, G^*)

is primitive. Thus, it satisfies the structure theory given in [31, Section 7]. We use this in Section 5.3.

Lemma 5.7 *Let $(X,G) \in \mathcal{S}$ be a definably primitive permutation group with* $\mathrm{rk}(G) > \mathrm{rk}(X) > 0$. *Then*

(i) G has no finite conjugacy classes;

(ii) $\mathrm{Soc}(G)$ *is definable.*

Proof. (i) Suppose there is a non-trivial finite conjugacy class. Then G has a definable characteristic subgroup N consisting of the finite conjugacy classes. By primitivity, N must act transitively on X. As N is a BFC group, $N' \lhd G$ is a finite abelian group and so by primitivity it must be the identity. So N is abelian and acts transitively and faithfully on X, and hence regularly.

Let (X^*, G^*, N^*) be an $\aleph_1$-saturated elementary extension of (X, G, N). Then G^* acts definably primitively on X^*, and hence primitively by Lemma 2.7. Let $x \in N^*$. Then x^{G^*} is finite, so $\langle x^{G^*} \rangle$ is a countable normal subgroup of G^* contained in N^*. Again, by saturation X^* is uncountable. So $\langle x^{G^*} \rangle$ cannot act transitively on X^*, which contradicts the primitivity of (X^*, G^*).

(ii) This is immediate from (i) and [20, Lemma 5.3(2)]. □

Lemma 5.8 *Let $(X,G) \in \mathcal{S}$ be a non-principal ultraproduct of finite permutation groups of the form $(X,G) = \prod_{j \in J}(X_j, G_j)/\mathcal{U}$, with* $\mathrm{rk}(G) > \mathrm{rk}(X)$.

(i) The permutation group (X,G) is primitive if and only if the permutation group (X_j, G_j) is primitive for ultrafilter-many $j \in J$.

(ii) Suppose (X,G) is primitive. Let $S(x)$ be a formula (guaranteed by Lemma 5.7) defining the socle of G. Then for ultrafilter-many $j \in J$ the formula $S(x)$ defines the socle in G_j.

Proof. i) $\Rightarrow$ If (X_j, G_j) is imprimitive for ultrafilter-many j, then the ultraproduct of the proper non-trivial G_j-congruences is a proper G-congruence on X.

$\Leftarrow$ If ultrafilter-many of the (X_j, G_j) are primitive, then (X,G) is definably primitive, and hence primitive by Lemma 2.7.

(ii) First observe that, by the proof of the O'Nan-Scott Theorem (see e.g. [15, Theorem 4.3B]), if (Y,H) is a finite primitive permutation group, then either H has a unique minimal normal subgroup, or $\mathrm{Soc}(H)$ is the direct product of the two minimal normal subgroups of H. In the latter case the two minimal normal subgroups of H both act regularly on Y.

By Lemma 5.7 and [20, Lemma 5.3(2)], $\mathrm{Soc}(G)$ is the direct product of finitely many minimal normal subgroups $N_1, \ldots, N_t$, each definable (by Lemma 3.5) by a formula $N_i(x)$. There is a sentence in $\mathrm{Th}((X,G))$ expressing that $S(G) = N_1(G) \times \ldots \times N_t(G)$.

By (i), we may suppose that each (X_j, G_j) is primitive. For ultrafilter-many j the formula $S(x)$ defines a normal subgroup of G_j contained in $S(G_j)$. For each $i = 1, \ldots, t$ and for ultrafilter-many j, the formula $N_i(x)$ defines a normal subgroup of G_j, and indeed for ultrafilter many j this is a *minimal* normal subgroup of G_j, since otherwise an ultraproduct of smaller normal subgroups will be a normal subgroup of G properly contained in N_i. Also, for ultrafilter-many j we have $N_1(G_j) \times \ldots \times N_t(G_j) = S(G_j)$, since this is first order expressible. Thus, for ultrafilter-many j, $S(G_j)$ is a normal subgroup of G_j contained in $\mathrm{Soc}(G_j)$, and (by the first paragraph) $t \le 2$.

It remains to check that, for ultrafilter-many j, the containment $S(G_j) \le \mathrm{Soc}(G_j)$ is not proper, so suppose that it is. Thus, we may suppose that for each j, G_j has a minimal normal subgroup M_j disjoint from $S(G_j)$. In this case, by the first paragraph, $t = 1$. Also, by the first paragraph, M_j acts regularly on X_j, so the ultraproduct M of the M_j acts regularly on X, and is a normal subgroup of G disjoint from $S(G)$ (as this is first order expressible in a language with an additional predicate for the M_j). Since M is a regular normal subgroup of a primitive group, M is minimal normal in G, so $M \le \mathrm{Soc}(G) = S(G)$. This is a contradiction. □

By the last lemma, primitive permutation groups $(X, G) \in \mathcal{S}$ may be investigated via the O'Nan-Scott theorem for finite primitive permutation groups. The statement below of this theorem is taken from [15, Theorem 4.1A]. More detail can be found in [32].

Theorem 5.9 *Let (X, G) be a finite primitive permutation group of degree n, and let H be the socle of G. Then either*

(a) H is a regular elementary abelian p-group for some prime p, $n = p^m = |H|$, and G is isomorphic to a subgroup of the affine group $\mathrm{AGL}_m(p)$ with its natural action on H (the latter identified with X); or

(b) H is isomorphic to a direct power T^m of a nonabelian simple group T and one of the following holds:

(i) $m = 1$ and G is isomorphic to a subgroup of $\mathrm{Aut}(T)$;

(ii) $m \ge 2$ and G is a group of "diagonal type" with $n = |T|^{m-1}$;

(iii) $m \ge 2$ and for some proper divisor d of m and some primitive permutation group (Y, K) with $\mathrm{Soc}(K)$ isomorphic to T^d, G is isomorphic to a subgroup of the wreath product K wr $\mathrm{Sym}(m/d)$ with the 'product action' on $Y^{m/d}$ and $n = (|Y|)^{\frac{m}{d}}$;

(iv) $m \ge 6$, H is regular, and $n = |T|^m$ (the 'twisted wreath' case).

In the proof of Theorem 5.1, we may assume that (X, G) is an ultraproduct of finite primitive permutation groups (X_j, G_j) which are all of one type from (a), (bi), (bii), (biii) or (biv). We define the type of (X, G) to be the uniform type of the (X_j, G_j). In particular, (X, G) has type (i) if and only if $\mathrm{Soc}(G)$ is abelian.

Suppose that $(X, G) \in \mathcal{S}$ is primitive, with $\mathrm{rk}(G) > \mathrm{rk}(X) = 1$, and $A := \mathrm{Soc}(G)$ is abelian. By Lemma 5.7, G has no non-trivial finite conjugacy classes and $\mathrm{Soc}(G)$ is definable. Then $\mathrm{Soc}(G)$ is elementarily abelian or torsion-free divisible abelian, by Lemma 5.2, so can be viewed as a vector space over a prime field. Also, as in Lemma 5.2 and the finite O'Nan-Scott Theorem, after identifying some $x \in X$ with $0 \in A$ and each element $a(x) \in X$ with $a \in A$, we may identify X with A (acting on itself by translation) and G with AH; here H is the stabiliser of the zero of A, its action on A is by conjugation, and under this action $H \leq \mathrm{GL}(A)$ is irreducible. Correspondingly, if (X,G) is a non-principal ultraproduct $\Pi_{j\in J}(X_j,G_j)/\mathcal{U}$ of finite permutation groups, we may suppose that for each $j\in J$ we have $G_j = A_jH_j$ with A_j elementary abelian and H_j an irreducible subgroup of $\mathrm{GL}(A_j)$. For any definable subgroup $Y\leq G$ we write Y_j for the corresponding subset of G_j (it is a group for ultrafilter-many j).

In the next lemma we use Clifford's Theorem (see e.g. [22, p.90], or [1, 12.13]). The basic assertion is that if V is a finite dimensional vector space over a field F, and $T \leq \mathrm{GL}(V)$ is irreducible, and $R \lhd T$, then we may write $V = V_1 \oplus \ldots \oplus V_l$ where $T \leq \mathrm{GL}(V_1)\mathrm{wr}\,\mathrm{Sym}_l$ acts naturally, and each V_i is a homogeneous FR-module, that is a module of the form $\langle W' : W' \leq V, W' \cong W\rangle$ (isomorphism of FR-modules), for some simple FR-submodule W of V. We refer to the V_i as the *Wedderburn components* of the FT-module V.

Notice that if $T\leq \mathrm{GL}(A)$, then the affine group AT acts primitively on A if and only if A is an irreducible T-module.

Lemma 5.10 *In the above notation, suppose that $(X,G)\in\mathcal{F}$ is definably primitive of affine type, with $G = AH$ as above. Assume that $1 = \mathrm{rk}(X) < \mathrm{rk}(G)$, and that A is an elementary abelian p-group. Suppose $B \leq H$, B is definable and A is an irreducible $\mathbb{F}_pB$-module. Let C be an infinite, definable normal subgroup of B. Then A is an irreducible $\mathbb{F}_pC$-module.*

Proof. We suppose for a contradiction that A is a reducible $\mathbb{F}_pC$-module.

Claim 1. (i) Let U be an $\mathbb{F}_pC$-submodule of A with no definable proper non-trivial submodules. Then U is definable.

(ii) There is a *definable* proper non-trivial $\mathbb{F}_pC$-submodule of A.

Proof of Claim. (i) We may suppose that U is infinite, as otherwise it is definable. Let $u\in U\setminus\{0\}$. We may also suppose u^C is infinite, as otherwise, since U is an $\mathbb{F}_p$-vector space, $\langle u^C\rangle$ is a finite, so definable, C-invariant non-trivial proper $\mathbb{F}_pC$-submodule of U, contrary to assumption. By Theorem 2.2, there is a definable C-invariant group W such that $W\leq \langle u^C\rangle$, and u^C/W is finite. Then W is a non-trivial definable C-invariant subspace of U, and it follows that $W = U$.

(ii) Since A is reducible, there is a proper non-trivial $\mathbb{F}_pC$-submodule U. By (i), either U is definable, or it contains a proper non-trivial definable $\mathbb{F}_pC$-submodule.

It follows in particular from Claim 1 that for ultrafilter-many $j\in J$, C_j acts reducibly on A_j. (This also follows from 'definably primitive implies primitive').

Next, we examine the consequences of Clifford's Theorem on the (finite) groups $C_j \lhd B_j \leq \mathrm{GL}(A_j)$.

Claim 2: There is a fixed positive integer t such that for ultrafilter-many $j\in J$, the $\mathbb{F}_pB_j$-module A_j has exactly t Wedderburn components.

Proof of Claim: Suppose the number of Wedderburn components were increasing unboundedly over the ultraproduct members. Thus for any $n\in\mathbb{N}$ there is an ultrafilter set $J_n \subseteq J$ where for all $j\in J_n$, $C_j \lhd B_j$ and B_j has more than

n Wedderburn components. Thus suppose such an A_j has some set of Wedderburn components $V_1, V_2, \dots, V_n$. For each $1 \le k \le n$ select $v_k \in V_k \setminus \{0\}$. Let $w_k = \sum_{j=1}^{k} v_j$. Clifford theory shows that for $1 \le k \ne l \le n$, the vectors w_k and w_l are not in the same B_j-orbit. By our assumptions, there would be infinitely many B-orbits on A. However, this is impossible. For since B acts irreducibly on the $\mathbb{F}_p$-vector space A, any non-trivial B-orbit on A is infinite, so as $\mathrm{rk}(A) = 1$, there are just finitely many B-orbits on A.

We now assume that there is a constant number t of Wedderburn components for all $j \in J$. Note that if the Wedderburn components are definable then $t = 1$: indeed, otherwise A is a direct sum of $t > 1$ infinite $\mathbb{F}_p$-subspaces $W^{(i)}$, and we have $0 < \mathrm{rk}(W^{(i)}) < \mathrm{rk}(A)$, contradicting our assumption that $\mathrm{rk}(A) = 1$. We may write $A_j = W_j^{(1)} \oplus \ldots \oplus W_j^{(t)}$, with $B_j \le \mathrm{GL}(W_j^{(1)})$ wr $\mathrm{Sym}(t)$.

Claim 3: There is a positive integer s such that, for almost all j, each Wedderburn component is a direct sum of exactly s $\mathbb{F}_p C_j$-irreducibles.

Proof of Claim: Again, we suppose not. Then for any positive integer n, it follows that for almost all j, each Wedderburn component $W_j^{(i)}$ in A_j is a direct sum of more than n irreducible C_j-subspaces, each isomorphic to $U_j^{(i)}$, say. The ultraproduct $U^{(i)}$ of the $U_j^{(i)}$ is an $\mathbb{F}_p C$-module with no definable proper non-trivial submodules, so is definable (by Claim 1(ii)), and it follows by rank considerations that $U^{(i)}$ is finite, so we may suppose the $U_j^{(i)}$ have a fixed finite size, which, by Clifford theory, does not vary with i. Hence, the kernel of the action of C_j on $U_j^{(i)}$, and hence on $W_j^{(i)}$, has a fixed finite index. It follows that the groups C_j have a fixed finite order, contradicting the assumption that C is infinite.

Thus, for each j, each Wedderburn component is a direct sum of boundedly many C_j-irreducibles. Hence, Wedderburn components are uniformly definable, by Claim 1. Also, Wedderburn components are conjugate, so have the same size, so as (for ultrafilter-many j) there are exactly t of them, for any n Wedderburn components have size at least n for ultrafilter-many j. It follows immediately (as $\mathrm{rk}(A) = 1$) that (for ultrafilter-many j) there is a unique, C_j-irreducible Wedderburn component of A_j. This proves the lemma. □

5.2 $\mathrm{rk}(G) = 2$

Proposition 5.11 *Let $(X, G) \in \mathcal{F}$ be a definably primitive pseudofinite permutation group, and suppose that $\mathrm{rk}(X) = 1$ and $\mathrm{rk}(G) = 2$. Then conclusion (ii) of Theorem 5.1 holds.*

Proof. The proof proceeds in a series of claims.

Claim 1. The group G contains a normal subgroup S of finite index which is soluble, and such that S' has rank 1 and contains an abelian regular normal subgroup $A \lhd G$.

Proof of Claim 1. By Theorem 1.2, we know that G is definably a soluble-by-finite group. Let T be such a definable, normal, soluble subgroup of G of finite index. Since T is soluble, there is m such that $T^{(m)} = \{1\}$, so there is a least $n \in \mathbb{N}$ such that $\mathrm{rk}(T^{(n+1)}) < 2$. Let $S = T^{(n)}$. By Corollary 2.4, S is definable. Now by our choice of S, it has finite index in G, and S' is of rank 1 or 0. Since S is normal in G, so is S'. If $\mathrm{rk}(S') = 0$ then $S' = \{1\}$ (by Lemma 5.7), so S must be abelian, composed only of elements of finite G-conjugacy classes, a contradiction to Lemma 5.7. So $\mathrm{rk}(S') = 1$. From Theorem 2.11, we know that S' has a definable characteristic subgroup A of finite index which is finite-by-abelian (and BFC). Thus the bottom finite part may be taken to be A', so characteristic in S', so normal in G, so trivial. Thus S' has a finite index abelian subgroup A, which is characteristic in S'. Thus A is normal in G, and so is transitive on X. In particular, A is regular.

It follows from Claim 1 that (X, G) is of affine type as in Lemma 5.2. We therefore write $G = AH$, adopting the notation $G_i = A_iH_i$ used earlier, with (X, G) an ultraproduct of finite permutation groups (X_i, A_iH_i), each of affine type as in Theorem 5.9(a). We view A as a vector space over a prime field F. The group A_i is an elementary abelian p_i-group, so may be viewed as an $\mathbb{F}_{p_i}$-vector space, with $H_i \leq \mathrm{GL}(A_i)$ irreducible.

Claim 2. There is $f : \mathbb{N} \to \mathbb{N}$ such that if Q is a finite group and R is a group of exponent n acting semi-regularly on Q, then $|R| < f(n)$.

Proof of Claim 2. See [1, 40.6].

Claim 3. The group H has a definable abelian normal subgroup Y of finite index such that A is an irreducible FY-module.

Proof of Claim 3. There are two cases, according to whether or not $\mathrm{char}(F) = 0$.

Suppose first that $\mathrm{char}(F) = 0$, so p_i is not constant on any ultrafilter-set. Let $a \in A \setminus \{0\}$, and suppose $h \in H$ fixes a. Then $C_A(h)$ is a definable $\mathbb{Q}$-subspace of A, so is infinite, so has finite index in A (as $\mathrm{rk}(A) = 1$); hence, as A is divisible, $C_A(h) = A$. As H acts faithfully on A, it follows that $h = 1$, that is, H acts semi-regularly on A. Thus, we may suppose that H_i acts semi-regularly on A_i for almost all i. Since H is infinite, and (by Claim 2) the exponent of the H_i increases with $|H_i|$, it follows that H has infinite exponent. In particular, by ω-saturation of ultraproducts there is $h \in H$ of infinite order. Now $C := C_H(C_H(h))$, which equals $Z(C_H(h))$, is an abelian definable subgroup of H which contains $\langle h \rangle$ so is infinite. As $\mathrm{rk}(H) = 1$, $|H : C|$ is finite, and $Y := \bigcap(C^k : k \in H)$ is a definable finite index abelian normal subgroup of H. If $v \in A \setminus \{0\}$ then v^Y contains an infinite definable Y-invariant subspace of A, by Theorem 2.2. Since F has characteristic 0 and $\mathrm{rk}(A) = 1$, this subspace equals A. Since v is arbitrary, Y is irreducible on A.

Next, suppose that $\mathrm{char}(F) = p$, a prime. Then we may suppose $p_i = p$ for all i. Now by Lemma 5.10, any infinite definable normal subgroup of H is irreducible on A, so to prove the claim it suffices to show that H has an infinite definable abelian subgroup (for this will have finite index in H, so we can then take Y to be the intersection of its conjugates). Arguing by contradiction, we may suppose H has finite exponent, as otherwise we may choose a double centraliser C as above. By Lemma 5.10, we may replace H by any definable normal subgroup of finite index. Therefore, using Theorem 2.11 and dropping to a finite index subgroup of H if necessary, we may suppose that H' is a finite subgroup of $Z(H)$; so in particular H is nilpotent of class 2. We view H as a permutation group on A, so for $a \in A$ we write H_a for $C_H(a)$. By Lemma 5.5, H has finitely many orbits on $A \setminus \{0\}$, say $U_1, \ldots, U_r$. For each $i = 1, \ldots, r$ let $a_i \in U_i$. Since H acts irreducibly on A and $F = \mathbb{F}_p$ is finite, each a_i^H is infinite, so H_{a_i} cannot have finite index in H so must be finite. Then, as H' is finite, $H'H_{a_i}$ is a finite normal subgroup of H. Put $L_i := C_H(H'H_{a_i})$. Then for each i, L_i is an infinite definable normal subgroup of H, so $|H : L_i|$ is finite. Put $L := L_1 \cap \ldots \cap L_r$, a definable normal subgroup of H. Also $|H : L|$ is finite and, by Lemma 5.10, L is irreducible on A. Also, $L_{a_i} \leq Z(L)$, since $L \leq C_H(H_{a_i})$. So as $L \lhd H$, $L_a \leq Z(L)$ for all $a \in U_i$, so for all $a \in A \setminus \{0\}$. As L acts faithfully on each orbit a^L (by irreducibility), it follows that $L_a = 1$ for all non-zero $a \in A$, that is, L acts semi-regularly on $A \setminus \{0\}$. Since H has finite exponent, so does L. It follows by Claim 2, applied to the corresponding finite groups L_i acting on A_i, that L is finite. Since $|H : L|$ is finite, this is a contradiction, so proves Claim 3.

We now apply Proposition 2.8 to the abelian normal subgroup Y of H of finite index, acting definably and F-irreducibly on A. So $K := \mathbb{Z}[Y]/\mathrm{ann}_{\mathbb{Z}[Y]}(A)$ is an interpretable pseudofinite field, and its additive group is identified with A. Now the action by conjugation of H on Y extends naturally to an action on $\mathbb{Z}[Y]$: for $\sum_{i=1}^{l}(-1)^{\varepsilon_i} y_i \in \mathbb{Z}[Y]$ and $h \in H$ we define $h(\sum_{i=1}^{l}(-1)^{\varepsilon_i} y_i)h^{-1} = \sum_{i=1}^{l}(-1)^{\varepsilon_i} h y_i h^{-1}$. This action preserves both the additive structure and multiplicative structure of $\mathbb{Z}[Y]$.

Claim 4. The action of H on $\mathbb{Z}[Y]$ fixes the ideal $\mathrm{ann}_{\mathbb{Z}[Y]}(A)$.

Proof of Claim. Suppose $y = \sum_{i=1}^{l}(-1)^{\varepsilon_i} y_i \in \mathrm{ann}_{\mathbb{Z}[Y]}(A)$. Then for any $a \in A$ we have $y \cdot a = 0$. Let $h \in H$ and let $x = hyh^{-1} = h(\sum_{i=1}^{l}(-1)^{\varepsilon_i} y_i)h^{-1}$. We must show that the endomorphism x maps a to 0. So we compute:

$$\begin{aligned}
x \cdot a \quad &= h(\sum_{i=1}^{l}(-1)^{\varepsilon_i} y_i)h^{-1} \cdot a \\
&= \sum_{i=1}^{l}(-1)^{\varepsilon_i} h y_i h^{-1} a h y_i^{-1} h^{-1} \\
&= h\big(\sum_{i=1}^{l}(-1)^{\varepsilon_i} y_i (h^{-1} a h) y_i^{-1}\big)h^{-1} \\
&= h(y \cdot (h^{-1} a h))h^{-1} \\
&= 0
\end{aligned}$$

We conclude that there is an action of H on the field K, induced from conjugation, that preserves its additive and multiplicative structure. In particular, H induces a group of automorphisms of K. The group Y embeds in the multiplicative group K^*.

Claim 5. H induces the trivial group on K.

Proof of Claim. Since $Y \leq C_H(K)$ and $|H : Y|$ is finite, $H/C_H(K)$ is a finite group of automorphisms of the pseudofinite field K, of order m, say. Now we consider that G is an ultraproduct of finite groups. So for ultrafilter-many $j \in J$, the formulas for K interpret a field K_j in the group G_j and the formulas for $H/C_H(K)$ interpret a group of automorphisms D_j of K_j of order m. So by Los's Theorem $H/C_H(K)$ is a cyclic group of order m. Furthermore, K_j must be definably an m-dimensional vector space over the fixed field E_j of a generator of D_j. In the ultraproduct this would mean that $\mathrm{rk}(K) = m\,\mathrm{rk}(E) \geq m$; for clearly, as K is infinite, so is E, so $\mathrm{rk}(E) \geq 1$. Since K is in definable bijective correspondence with A we have $\mathrm{rk}(K) = 1$, and so we deduce that $m = 1$. So $H = C_H(K)$.

Finally, we show that we may assume $H = Y$, and that $|K^* : Y|$ is finite. Let $t, s \in H$, and let $a \in A\backslash\{0\}$. There are unique $c, d \in K^*$ such that $tat^{-1} = c \cdot a$ and $sas^{-1} = d \cdot a$. Suppose $c = \sum_{i=1}^{l}(-1)^{\varepsilon_i}c_i$ and $d = \sum_{i=1}^{l}(-1)^{\delta_i}d_i$ and so $tat^{-1} = \sum_{i=1}^{l}(-1)^{\varepsilon_i}c_iac_i^{-1}$ and $sas^{-1} = \sum_{i=1}^{l}(-1)^{\delta_i}d_iad_i^{-1}$. Then an easy calculation shows that $stat^{-1}s^{-1} = tsas^{-1}t^{-1}$. But the action by conjugation of H on A is faithful. Since the above computation is for arbitrary $a \in A$ it follows that $st = ts$. So H is an abelian group and hence in the above proof we may take $H = Y$. Since $\mathrm{rk}(K) = \mathrm{rk}(K^*) = 1$ and $\mathrm{rk}(G) = 2$, it follows that $|K^* : Y|$ is finite. It is now clear that AH satisfies the conclusion of Proposition 2.8. □

5.3 $\mathrm{rk}(G) \geq 3$

To prove Theorem 5.1, it remains for us to prove the following proposition.

Proposition 5.12 *Let $(X, G) \in \mathcal{F}$ be a definably primitive permutation group, and suppose that* $\mathrm{rk}(X) = 1$ *and* $\mathrm{rk}(G) \geq 3$. *Then there is a definable pseudofinite field K such that* $\mathrm{PSL}_2(K) \leq G \leq \mathrm{P\Gamma L}_2(K)$*, and G has the natural action on* $\mathrm{PG}_1(K)$ *(identified with X). Furthermore,* $|G : \mathrm{PSL}_2(K)|$ *is finite, so* $\mathrm{rk}(G) = 3$.

The remainder of this paper is a proof of Proposition 5.12. As in the case when $\mathrm{rk}(G) = 2$, we suppose that (X, G) is a non-principal ultraproduct $\Pi_{j\in J}(X_j, G_j)/\mathcal{U}$ of finite primitive permutation groups, and that all the (X_j, G_j) have the same type in the sense of the O'Nan-Scott Theorem, Theorem 5.9. The assumptions of Proposition 5.12 hold from now on.

We first eliminate the affine case. So suppose that $\mathrm{Soc}(G)$ is an abelian group A, that is, case (a) of 5.9 holds. As in the proof of Proposition 5.11, we have $G = AH$, where A is identified with X, and H is the stabiliser of the identity 0 of A. Again, we view A as a vector space over a prime field, and $H \leq \mathrm{GL}(A)$ is irreducible. Replacing J by a subset in the ultrafilter if necessary, we may suppose that there is a corresponding decomposition $G_j = A_j H_j$ for each $j \in J$.

Lemma 5.13 *The characteristic of A_j is constant on an ultrafilter set.*

Proof. Suppose not. Then since the A_j are uniformly definable, we deduce that A is a $\mathbb{Q}$-vector space. Now $H \leq \mathrm{GL}(A)$. Pick $a \in A \setminus \{0\}$. By the rank assumption $\mathrm{rk}(H_a) \geq 1$, so there is $h \in H_a \setminus \{1\}$. Since a has infinite order, $C_A(h)$ is an infinite definable subgroup of A, so $|A : C_A(h)|$ is finite. By the divisibility of A, h fixes the whole of A, contrary to the faithfulness of G on X. □

Thus, we may assume that A (and also the A_i) has the structure of a vector space over $\mathbb{F}_p$, with $H \leq \mathrm{GL}_\kappa(p)$ for some infinite cardinal κ.

We consider now a maximal chain of definable groups $1 = N_0 \triangleleft N_1 \ldots \triangleleft N_r = H$, with $\mathrm{rk}(N_i) < \mathrm{rk}(N_{i+1})$ for each $i = 0, \ldots, r-1$. By Lemma 5.10 and induction, N_i acts irreducibly on A for each $i > 0$.

Consider first N_1. Let B be the definable normal subgroup of N_1 consisting of its finite conjugacy classes.

Suppose first that B is infinite. In this case, B is a characteristic subgroup of N_1 of finite index, so we may replace N_1 by B, that is, we assume all H-conjugacy classes in N_1 are of finite (hence bounded) size. Thus, N_1 is finite-by-abelian, and we may suppose it is centre-by-abelian. By Lemma 5.10, N_1 acts irreducibly on A. Hence all N_1-orbits on A are infinite, so for non-zero $a \in A$, $\mathrm{rk}((N_1)_a) = \mathrm{rk}(N_1) - 1$. Thus, $(N_1)_a N_1'$ is a definable normal subgroup of N_1 of rank strictly less than $\mathrm{rk}(N_1)$, so by minimality of $\mathrm{rk}(N_1)$, it is finite. Hence, $\mathrm{rk}(N_1) = \mathrm{rk}((N_1)_a) + 1 = 1$. Since N_1 is irreducible on A, (A, AN_1) is a primitive permutation group with $\mathrm{rk}(AN_1) = 2$, and it follows by Proposition 5.11 that AN_1 is a subgroup of $\mathrm{AGL}_1(F)$ for some definable field F. Since $\mathrm{rk}(H) > 1$, N_2 exists and as F is a prime field we must have $AN_2 \leq \mathrm{A\Gamma L}_1(F)$ (this is standard – compare the proof of Proposition 5.11). Thus, as j ranges through J there is a uniformly definable finite field F_j with additive group A_j, and elements of $(N_2)_j$ induce field automorphisms which are arbitrarily large powers of the Frobenius. In particular, by taking fixed points, the A_j have definable subgroups of arbitrarily large order and index, contradicting that $\mathrm{rk}(A) = 1$.

Thus, B is finite. Then let y^{N_1} be any infinite conjugacy class of N_1. The Indecomposability Theorem (Theorem 2.2) shows that there is a definable, normal subgroup $N_y \triangleleft N_1$ with $N_y \subseteq \langle y^{N_1} \rangle$ and y^{N_1}/N_y finite. By our assumptions N_y must have finite index in N_1. Furthermore, the groups N_y are uniformly definable (compactness), so as there are no infinite descending chains of uniformly definable subgroups, N_1 has a minimal, definable normal subgroup of finite index. Since this is characteristic in N_1 we replace N_1 with this latter group. Now any definable normal subgroup of N_1 is of rank less than $\mathrm{rk}(N_1)$, so is finite. It follows by Proposition 3.4 that either N_1 is soluble, or $R(N_1)$ is finite and $N_1/R(N_1)$ is a product of finitely many definable finite or pseudofinite non-abelian simple groups, with just one pseudofinite one. If N_1 is soluble, then as N_1' is definable (by Corollary 2.4) it is finite, so N_1 is a BFC group, contrary to our assumption.

We have reduced to the case when $N_1/R(N_1) = M_1 \times M_2$, where M_1 is a simple pseudofinite group and M_2 is a product of finitely many finite simple groups. Let $\overline{M_i}$ be the preimage of M_i in N_1, for each $i = 1, 2$. The groups $\overline{M_i}$ are definable. By replacing N_1 by $C_{\overline{M_1}}(R(N_1))$ (a subgroup of finite index), we reduce to the case when N_1 is a finite-centre-by-simple and perfect, so quasisimple. Notice also, by our knowledge of rank 1 groups in $\mathcal{M}$ (Theorem 2.11), that $\mathrm{rk}(N_1) > 1$. Since the group $N_1/Z(N_1)$ is an infinite pseudofinite simple group, it is a group of Lie type, possibly twisted, over a pseudofinite field. Since A is an irreducible $\mathbb{F}_p N_1$-module (by Lemma 5.10), the affine permutation group (A, AN_1) is primitive. Hence, by Lemma 5.5, for each $a \in A \setminus \{0\}$ the point stabiliser $(N_1)_a$ has finitely many orbits on A. We claim that there is a field F extending $\mathbb{F}_p$ such that $AN_1 \leq \mathrm{A\Gamma L}_n(F)$, i.e., that A has the structure of a *finite-dimensional* vector space over some field F, with $N_1 \leq \Gamma\mathrm{L}_n(F)$. This follows from the description of infinite dimensional affine permutation groups with finitely many orbits on pairs in [31] (see Proposition 3.6, and also Section 7). In fact, since N_1 is quasisimple, easily $N_1 \leq \mathrm{GL}_n(F)$. Likewise, again via [31, Proposition 3.6], we may suppose that $A_j(N_1)_j \leq \mathrm{AGL}_n(F_j)$. Note that the finite fields F_j have unbounded size.

The following claim now eliminates the affine case in Proposition 5.12.

Claim. Suppose that $(X, G) \in \mathcal{F}$ satisfies all the reductions above. Then $\mathrm{rk}(X) > 1$.

Proof of Claim. Suppose for a contradiction that $\mathrm{rk}(X) = 1$, that (X, G) is an ultraproduct as above, and the above reductions and notation apply. By Lemma 5.13, A has prime characteristic p. For ultrafilter-many $j \in J$, A_j is over a finite field of characteristic p. We suppose that H is a Chevalley group $\mathrm{Chev}(K)$. By Lemma 5.5, G has finitely many orbits on X^2. So by Remark 5.6, the assumptions of Section 3 of [31] apply.

We claim that H_j has the same characteristic p as A_j. By the main theorem of Landazuri and Seitz [30], if $H_j = \mathrm{Chev}(q_j)$ and $\mathrm{char}(H_j) \neq p$ then the least possible dimension of an irreducible characteristic p representation of $\mathrm{Chev}(q_j)$ increases with q_j. Thus, we would be in the 'unbounded dimension' case of [31], handled by Proposition 3.6 there, and the cross-characteristic case does not arise.

Thus, H_j has the form $\mathrm{Chev}(p^{a_j})$ for some $a_j \in \mathbb{N}$. Let P_j be a Sylow p-subgroup of H_j, that is, a maximal unipotent subgroup. It is easily checked that the P_j are uniformly definable, as j varies in J. (This follows since maximal unipotent subgroups of finite simple groups of Lie type are uniformly definable, by [7, 5.3.3(ii), 13.6.1].) Since non-trivial P_j-orbits on A_j have size a power of p, and P_j fixes the zero vector so acts on $A_j \setminus \{0\}$ which has size $p^{n_j} - 1$ for some n_j, by divisibility considerations P_j has a non-trivial fixed point on A_j. Thus, $C_{A_j}(P_j)$ is a definable non-trivial subspace of A_j, proper by faithfulness. Since A is finite-dimensional and $|F|$ is infinite, it follows that $\mathrm{rk}(A) \geq 2$, completing the proof of the claim.

Next, we eliminate types (b) (ii)-(iv) of Theorem 5.9.

Lemma 5.14 *If* $(X, G) \in \mathcal{F}$ *is a primitive permutation group with* $\mathrm{rk}(X) = 1$*, then for ultrafilter many* $j \in J$*,* (X, G) *is not of diagonal type, product action type, or twisted wreath type of the O'Nan-Scott Theorem.*

Proof. In the diagonal case, by Lemma 5.1 of [31] we may suppose that $\mathrm{Soc}(G) = T^k$ for some definable non-abelian pseudofinite simple group T, and that X may be identified with the coset space in T^k of a diagonal subgroup of T^k. Thus, the rank of X equals the rank of T^{k-1}, which cannot be 1.

If (X, G) is of product action type, then there is a definable primitive infinite permutation group (Y, H) such that X is in definable bijection with Y^l for some integer $l > 1$. Thus, $\mathrm{rk}(X) = l\,\mathrm{rk}(Y) > 1$, a contradiction.

Finally, Lemma 5.3 of [31] eliminates the twisted wreath case. □

To complete the proof of Proposition 5.12, it remains to handle the case of non-abelian simple socle, that is, to prove the following.

Lemma 5.15 *Suppose* (X, G) *is a definable primitive permutation group in* $\mathcal{F}$*, that* $\mathrm{rk}(X) = 1$*, and that* $\mathrm{Soc}(G)$ *is a non-abelian simple group. Then there is a pseudofinite field* F *such that* $\mathrm{PSL}_2(F) \leq G \leq P\Gamma L_2(F)$*, in its natural action on 1-spaces of* $\mathrm{PG}_1(F)$*. Furthermore,* $\mathrm{rk}(G) = 3$*.*

Proof. The permutation group (X, G) is an ultraproduct of finite primitive permutation groups (X_j, G_j). By Lemma 5.5, G_x has finite number, say r, of orbits on X, and the same statement holds for ultrafilter-many $j \in J$. The group $\mathrm{Soc}(G)$ is a pseudofinite simple group, so is a Chevalley group (possibly twisted) over a pseudofinite field. It follows that there is a fixed Lie type $L(q)$ such that for ultrafilter-many $j \in J$, $\mathrm{Soc}(G_j)$ has Chevalley type $L(q)$. In particular, as the Lie rank of $L(q)$ is fixed, q is unbounded. By the main theorem of Seitz [41], it follows that the action of G_j on X_j is parabolic (i.e. is on the cosets of a maximal parabolic subgroup P_j) for almost all $j \in J$. Thus, it suffices to show that the only possibility for a coset space of a parabolic subgroup to have rank 1 is the projective line.

We may suppose that $G_j = \mathrm{Soc}(G_j)$ for all j, since the coset space $\mathrm{Cos}(G_j : P_j)$ is in definable bijection with $\mathrm{Cos}(\mathrm{Soc}(G_j) : \mathrm{Soc}(G_j) \cap P_j)$. We remark that P_j is (uniformly) definable in G_j. This is proved in [14, Lemma 6.4], and follows rapidly from the uniform definability of maximum unipotent subgroups (see the proof of the last claim) and the Bruhat decomposition.

Let P be the ultraproduct of the P_j. We may identify X with $\mathrm{Cos}(G : P)$. Then G is bi-interpretable (in fact, bi-definable) with a rank 1 field or difference field, and the same holds, uniformly, for ultrafilter-many of the G_j. On an ultraproduct of finite fields, by the results in [9] (see also [10, Section 3]), the rank of a definable set X is determined by the approximate cardinalities of the corresponding X_j. By bi-definability, this is transferred to G and the G_j. That is, if G_j is bi-interpretable with $\mathbb{F}_{q_j}$ then an ultraproduct of uniformly definable sets each of cardinality roughly μq_j^d has rank d. Thus, using that the P_j are uniformly definable in the G_j, we must verify the following claim.

Claim. Let $G(q)$ be a group of Lie type (possibly twisted) over a field $\mathbb{F}_q$, with $G \neq \mathrm{PSL}_2(q)$, and let $P(q)$ be a parabolic subgroup of $G(q)$. Then $|G(q) : P(q)| > O(q)$.

Proof of Claim. If $G(q)$ is a classical group, then this follows from the main theorem of [12], which determines the permutation representations of the classical groups of minimal degree. (Recall that the *degree* of a permutation group (X, G) is $|X|$.)

So we may suppose that G is an exceptional group. We may also suppose that the Lie rank of $G(q)$ is at most 2. For otherwise, $G(q)$ has a simple subgroup $H(q)$ of Lie rank 2 with $H(q) \not\leq P(q)$. Since $|G(q) : P(q)| \geq |H(q) : P(q) \cap H(q)|$, we may then replace $G(q)$ by $H(q)$.

The degrees of the parabolic permutation representations of ${}^2B_2(q)$ and ${}^2G_2(q)$, which are 2-transitive, are listed in the proof of Theorem 5.3 [6], and are of order $O(q^2)$ and $O(q^3)$, respectively. Those of the groups ${}^2F_4(2^{2n+1})$, ${}^3D_4(q)$ and $G_2(q)$ correspond to the numbers of points and lines in the corresponding generalised polygons: here, the point set and line set of the polygon are coset spaces of appropriately chosen parabolics, two cosets incident if and only if not disjoint. A generalised polygon has associated parameters s and t, where there are $s+1$ points incident with each line and $t+1$ lines incident with each point. By [27, A,4], the values of (s,t) for the polygons of type G_2, ${}^3D_4(q)$ and ${}^2F_4(q)$ are respectively (q,q), (q,q^3) and (q,q^2). In [35, 1.5.4] there are formulas giving the numbers of points and lines in a generalised polygon in terms of the parameters s,t. From this data, it is easy to see that for parabolic subgroups $P(q)$ of these groups $G(q)$, we have $|G(q) : P(q)| \geq O(q^2)$, as required.

Given the claim, we know that $\mathrm{PSL}_2(F) \leq G \leq \mathrm{P\Gamma L}_2(F)$, and G has the natural action on the projective line. To check that $\mathrm{rk}(G) = 3$, we must show that $|G : \mathrm{PSL}_2(F)|$ is finite. If this is false, there is a pseudofinite supersimple finite rank structure consisting of a pair (F, B) where F is a field and B is an infinite group with a definable faithful action on F as a group of automorphisms. It is easy to see that there is $b \in B$ such that $\mathrm{Fix}(b)$ is an infinite field and $\mathrm{Fix}(b) < F$ is an infinite degree extension. This contradicts that $\mathrm{rk}(F)$ is finite. □

6 Further Observations

We note here two results related to Questions 3 and 4 from the Introduction. The first is an addendum to Theorems 1.2 and 1.3. We would like to prove it without (CFSG).

Proposition 6.1 *Let $G \in \mathcal{F}$ be a simple group of rank 3. Then $G \cong \mathrm{PSL}_2(F)$ for some pseudofinite field F.*

Proof. By Proposition 2.14, G is a group of Lie type.

First note that by Theorem 1.2 and 2.11, there is no infinite simple group in $\mathcal{F}$ of rank less than 3. Hence, we may suppose that G has Lie rank 1. Indeed, otherwise G has a parabolic subgroup $P = UL$, where L itself contains a simple group of Lie type over the same field, is definable, and has infinite index in G so has lower rank. (Here, L is definable, since it is a product of a maximal torus and a bounded number of root groups, all definable by results from [39, Chapter 5]; see also [7, Section 8.5]. The simple group is L', so is also definable.)

Thus, it suffices to show that ultraproducts of Suzuki groups ${}^2B_2(2^{2n+1})$ and Ree groups ${}^2G_2(3^{2n+1})$ have rank greater than 3. By [39, Proposition 5.4.6], these groups are uniformly bi-interpretable (over parameters) with difference fields $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$ or $(\mathbb{F}_{3^{2k+1}}, x \mapsto x^{3^k})$ respectively. The ultraproduct of the difference field has rank 1. It follows that, putting $q := 2^{2k+1}$ (or $q = 3^{2k+1}$ for the Ree groups), uniformly definable sets in the fields of cardinality roughly μq^d yield definable sets in the ultraproduct of rank d. It follows by consideration of orders (see e.g. [23, p. 135]) that an ultraproduct of finite Suzuki groups has rank 5, and an ultraproduct of finite Ree groups of type ${}^2G_2$ has rank 7. □

Theorem 6.2 *Let $(X, G) = \Pi_{J \in J}(X_j, G_j)/\mathcal{U}$ be a non-principal ultraproduct of finite primitive permutation groups, with $(X, G) \in \mathcal{F}$. Then there is $U \in \mathcal{U}$ and some $c \in \mathbb{N}$, such that if $j \in U$ then $|G_j| \leq |X_j|^c$.*

Proof. If point stabilisers G_x are finite, then the result is obvious, so we may suppose that for $x \in X$ we have $\mathrm{rk}(G_x) \geq 1$, so (X, G) is primitive by Lemma 2.7. Thus, the structure theory of [31], and in particular Section 7, applies. We may suppose that all the finite permutation groups (X_j, G_j) are primitive, and all have the same type in the sense of the O'Nan-Scott Theorem. In the affine case, by supersimplicity and [31, Theorem 1.1] it is easy to check that there is a natural number n, fixed on an ultrafilter set, such that X_j can be identified with a vector space $V_n(q)$ with $G_j \leq \mathrm{A\Gamma L}_n(q)$. (Observe here that a family of primitive permutation groups $(X_j, G_j) = (V_n(q), \mathrm{A\Gamma L}_n(q))$, where $n \to \infty$ as $j \to \infty$, cannot have a supersimple nonprincipal ultraproduct; for $\mathrm{A\Gamma L}_n(q)$ has a chain of uniformly definable subgroups – namely stabilisers of linearly independent tuples of vectors – with successive indices arbitrarily large.) Thus, $|X| = q^n$ and $|G| \leq q^{n^2+2}$, so it suffices to choose c so that $cn \geq n^2 + 2$. In the almost simple case, again by [31, Theorem 1.1.1] we find that for some n, $\mathrm{Soc}(G_j)$ has fixed Lie rank n. In this case, the result follows from the main theorem of [2]. The product action and diagonal action cases are also easily handled, either directly or using [2]. □

Remark 6.3 From Theorem 6.2 and the results in [31] it should be possible to answer the final question in the Introduction, showing that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that if $(X, G) \in \mathcal{F}$ is definably primitive then $\mathrm{rk}(G) \leq f(\mathrm{rk}(X))$. The idea is that (X, G) should essentially be an ultraproduct of an asymptotic class, uniformly bi-interpretable with a class of finite fields or difference fields, so that the asymptotic result in the last theorem should convert to a bound on the rank. Care is needed with how the exponent c provided by [2] varies with the Lie rank n. We have not verified the details.

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School of Mathematics; University of Leeds; Leeds LS2 9JT; UK; larry.elwes@gmail.com

Universite Lyon 1; INSA de Lyon, F-69621; Ecole Centrale de Lyon; CNRS, UMR5208, Institut Camille Jordan; 43 blvd du 11 novembre 1918; F-69622 Villeurbanne-Cedex; France; jaligot:@amth.univ-lyon1.fr

School of Mathematics, University of Leeds, Leeds LS2 9JT, UK; h.d.macpherson@leeds.ac.uk

65 Albion Gate; Albion St.; London W2 2LA; UK; mjryten@hotmail.com