Measuring Fixed Sets in SMA

Richard Elwes

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1 Introduction

These notes are essentially an expansion of the proof of Proposition 11.1 in [2]. The context is a strongly minimal theory T in a language L, which satisifies the DMP (definable multiplicity property) and has elimination of imaginaries. (Almost certainly elimination of Galois imaginaries is sufficient, but I haven't checked this yet.) Analogously to ACFA, we can adjoin a new unary function symbol σ to L to obtain L_σ, and we look at the model companion of the theory in which σ is interpreted as an automorphism. This model complete theory is called TA. (By [4] the DMP is exactly the criterion needed for TA to exist.)

We work in $(M, \sigma) \models TA$, which we will assume to be saturated. We will appeal to some elementary results from [5], in which many properties of ACFA are generalised to this context.

In these notes we are concerned with the fixed set F of σ. We know by Proposition 4.10 of [5], that any subset of F which is L_σ-definable in the structure (M, σ) with parameters from M, is in fact L-definable in the structure F (with parameters from F). Now, Proposition 11.1 in [2] states that F is *measurable*:

Definition 1.1. An infinite structure F is *measurable* if to every definable set X of F we can associate a pair $(\mathrm{Dim}(X), \mathrm{Meas}(X))$, where $\mathrm{Dim}(X)$ $\mathrm{Meas}(X) \geq 0$ such that the following hold.

1. For each L-formula $\varphi(\bar{x}, \bar{y})$ there is a finite set $D \subset \mathbf{N} \times \mathbf{Rr}^{>\mathbf{0}} \cup \{(\mathbf{0}, \mathbf{0})\}$, so that for all $\bar{a} \in M^m$ we have $(\mathrm{Dim, Meas})(\varphi(M^n, \bar{a})) \in D$.
2. If X is finite then $(\mathrm{Dim, Meas})(X) = (0, |\varphi(M^n, \bar{a})|)$.
3. For every L-formula $\varphi(\bar{x}, \bar{y})$ and all $(d, \mu) \in D_\varphi$, the set $\{\bar{a} \in M^m : h(\varphi(M^n, \bar{a})) = (d, \mu)\}$ is $\emptyset$-definable.
4. If X and Y are disjoint, then

$$(\mathrm{Dim, Meas})(X \sqcup Y) = \begin{cases} (\mathrm{Dim}(X), \mathrm{Meas}(X) + \mathrm{Meas}(Y)) \\ \qquad\qquad\qquad\qquad \text{if } \mathrm{Dim}(X) = \mathrm{Dim}(Y) \\ (\mathrm{Dim}(X), \mathrm{Meas}(X)) \text{ if } \mathrm{Dim}(X) > \mathrm{Dim}(Y) \\ (\mathrm{Dim}(Y), \mathrm{Meas}(Y)) \text{ if } \mathrm{Dim}(X) < \mathrm{Dim}(Y) \end{cases}$$

5. (Fubini) If $f : X \to Y$ is a definable surjection, where the fibres have constant dimension and measure (d, μ), then $(\mathrm{Dim}, \mathrm{Meas})(X) = (\mathrm{Dim}(Y) + d, \mathrm{Meas}(Y) \cdot \mu)$.

For further background on this notion, see [1]. In our case, the measure on F will be *normalized*: if X is L-definable in M over F and of Morley degree 1, then $\mathrm{Meas}(X(F)) = 1$.

Notation 1.2. We will write L_σ to mean $L \cup \{\sigma\}$, and acl^- or acl^σ, etc. to indicate the language being used.

If p and q are complete types, we will abuse notation by writing $f : p \to q$ to mean that f is function on the domains of p and q in M.

In all that follows $x, y, z, a, b, \ldots$ will be tuples in M.

2 Comments on G^{op}

If a group G acts on a set Ω, then G^{op} is by definition the automorphism group of the structure $(\Omega, g)_{g \in G}$, i.e the subgroup of $Sym(\Omega)$ which commutes with the action of G.

Lemma 2.1. *Suppose G acts faithfully on Ω and G^{op} is transitive on Ω. Then every point stabilizer of the action of G is trivial.*

Proof. Suppose $g \in G$ and $g(y) = y$. Then for any $\zeta \in G^{op}$ we have $g(\zeta(y)) = \zeta(g(y)) = \zeta(y)$. By the transitivity assumption it follows that $g(z) = z$ for all $z \in \Omega$. Thus $g = 1$. ■

Lemma 2.2. *If G acts regularly on Ω then G^{op} acts regularly too, and $G \cong G^{op}$, the isomorphism depending on choosing an element of Ω.*

Proof. With y fixed, for each $g \in G$ define $g^* : \Omega \to \Omega$ by $g^*(h(y)) := h(g(y))$. Then for any $h_1, h_2 \in G$ we have $g^*(h_1h_2(y)) = h_1h_2(g(y)) = h_1(g^*(h_2(y)))$. Thus for any $z \in \Omega$ and $h \in G$, we have $g^*(h(z)) = h(g^*(z))$, so $g^* \in G^{op}$. Of course the map $^* : G \to G^{op}$ is injective, and it is an easy exercise to check that it is a homomorphism, and depends on y. Given $\beta \in G^{op}$, there exists unique $g \in G$ such that $\beta(y) = g(y)$. Then for all $h \in G$, we have $\beta(h(y)) = h\beta(y) = hg(y)$. Thus for all $z \in \Omega$ we have $\beta(z) = g^*(z)$, and so * is surjective.

Note also that $g^*(y) = g(y)$, from which it follows that G^{op} acts transitively, and that G^{op} acts regularly: if $g^*(h(y)) = h(y)$ for any h, then $h(g(y)) = h(y)$, so $g(y) = y$, and $g = 1 = g^*$. ■

Comment 2.3. It follows from Lemmas 2.1 and 2.2 that if G is faithful and transitive on Ω, then both G and G^{op} act regularly.

3 Galois Maps

Definition 3.1. *Let p and q be a L-types over some $A \subseteq M$. Let $f : q \to p$ be an $L(A)$-definable map on the domains of these types. We say that f is* **Galois** *if there is a finite group G whose elements are $\emptyset$-definable, and which acts $\emptyset$-definably as a regular permutation group on $f^{-1}\{x\}$ for each $x \models p$.*

Notice that for any such f we may define: $q' := \mathrm{tp}(f(y), y)$ for $y \models q$, and $f' : q' \to p$ by projection onto the first coordinate, and then q is interdefinable with q' over A, and f with f', and f' is again Galois with group G. Hence we will assume that f is $L(\emptyset)$-definable.

The following is the canonical example of a Galois map:

Example 3.2. Suppose that r and p are complete L-types over some $A \subseteq M$, and $\tau : r \to p$ is an $L(A)$-definable finite-to-1 map. Then we pick $a \models p$, and choose an enumeration of $\bar{b}$ of $\tau^{-1}\{a\}$. Then set $q := \mathrm{tp}^{-}(a, \bar{b})/A$ and let $f : q \to p$ be projection onto the first co-ordinate.

Lemma 3.3. *Suppose $f : q \to_G p$ is Galois, and $\tau : q \to r$, and $\chi : r \to p$ are such that $f = \chi \circ \tau$. Put $H := \{h \in G | \forall y \models q, \quad \tau(hy) = \tau(y)\}$. Then $\tau : q \to_H r$ is Galois.*

Furthermore $H \triangleleft G$ if and only if $\chi : r \to p$ is Galois, in which case it has Galois group G/H.

Proof. Notice that, as q and r are complete types, $H := \{h \in G | \exists y \models q, \ \tau(hy) = \tau(y)\}$. Given $z \models r$ and $y_1, y_2 \in \tau^{-1}\{z\}$, there is a unique $h \in G$ where $h(y_1) = y_2$, since $y_1, y_2 \in f^{-1}\{\chi(z)\}$. Clearly $h \in H$.

Assuming now that $H \triangleleft G$, given $z_1, z_2 \in \chi^{-1}\{x\}$, pick $y_i \in \tau^{-1}\{z_i\}$. Then there is unique $g \in G$ where $g(y_1) = y_2$. Put $(gH)(z_1) = z_2$.

To see this is well-defined, pick $y'_i \in \tau^{-1}\{z_i\}$, and say that $g' \in G$ where $g'(y'_1) = y'_2$. Now there are $h_1, h_2 \in H$ where $h_i(y_i) = y'_i$. Then $g'(h_1(y_1)) = h_2(y_2)$, and $(h_2^{-1} g' h_1)(y_1) = y_2$. By regularity, it follows that $h_2^{-1} g' h_1 = g$, thus $g' h_1 = h_2 g = g \cdot g^{-1} h_2 g$, and so $g'H = gH$.

Also the action is regular: if $z_1, z_2 \in \chi^{-1}\{x\}$, then we've shown that gH where $(gH)(z_1) = z_2$ is unique.

Conversely if $\chi : r \to_J p$ is Galois, then define $\nu : G \to J$ by $\nu(g) \cdot \tau(y) := \tau(gy)$. Then ν is a homomorphism: $\nu(g_1 g_2) \cdot \tau(y) = \tau((g_1 g_2) y) = \nu(g_1) \cdot \tau(g_2 y) = \nu(g_1)\nu(g_2)\tau(y)$.

Also ν is surjective: let $j \in J$. Pick $z_0 \in \chi^{-1}\{x\}$, $y_0 \in \tau^{-1}\{z_0\}$, and $y_1 \in \tau^{-1}\{j(z_0)\}$. Then there is a unique $g \in G$ where $g(y_0) = y_1$, and $\nu(g)\tau(y_0) = \tau(gy_0)$, so $\nu(g)(z_0) = j(z_0)$. Thus by regularity $\nu(g) = j$.

Finally $g \in \mathrm{Ker}(\nu) \Leftrightarrow \forall y \ \ \nu(g) \cdot \tau(y) = \tau(y) \Leftrightarrow \forall y \ \ \tau(gy) = \tau(y) \Leftrightarrow \in H$. ■

Comment 3.4. If $f : q \to_G p$ is Galois over A, then writing $\Omega_x := f^{-1}\{x\}$ for $x \models p$, then the category whose object-set is $\{\Omega_x | \ \ x \models p\}$ and whose morphism-set is $\mathrm{Aut}(q(M)/A) := \{\alpha \restriction_{q(M)} | \alpha \in Aut(M/A)\}$ forms a concrete definable **groupoid**, see [3].

4 Artin Classes

If $f : q \to_G p$ is Galois and $x \in p(F)$, then the automorphism σ fixes $\Omega_x := f^{-1}\{x\}$ setwise, and thus acts as an element of G^{op} on Ω_x. As $G^{op} \cong G$, we find an element $g \in G$ corresponding to σ: of course g depends on the choice of isomorphism $G^{op} \cong G$, which in turn depends on picking an element $y \in \Omega$ (as in Lemma 2.2). However the conjugacy class g^G is fixed: say $\sigma(y_i) = g_i(y_i)$ where $i \in \{1, 2\}$. Then there is unique $h \in G$ where $h(y_1) = y_2$. Then $hg_1(y_1) = h\sigma(y_1) = \sigma(hy_1) = \sigma(y_2) = g_2(y_2) = g_2(h(y_1))$. Thus by regularity, $hg_1 = g_2h$, and $g_1 = h^{-1}g_2h$.

Definition 4.1. *We call g^G the* **Artin class** *of x, denoted $\mathrm{Artin}_f(x)$. We say that sets of the form $X = \{x \in p(F) | \mathrm{Artin}(x) \in S\}$ for some $S \subseteq G$ closed under conjugation are* **Artin-definable**.

Notice that an Artin-definable set X is definable relative to $p(F)$: it is the intersection of $p(F)$ with the definable set $\{x : \exists y \ (\sigma(y) = g(y)) \wedge (f(y) = x)\}$.

We obtain the following quantifier-elimination result.

Proposition 4.2. *Let $p(x)$ be a complete L-type over some $A \subseteq F$, and $a_1, a_2 \models p(x) \wedge \sigma(x) = x$. Then $\mathrm{tp}_\sigma(a_1/A) = \mathrm{tp}_\sigma(a_2/A)$ if and only if whenever $f : q \to_G p$ is Galois, then $\mathrm{Artin}_f(a_1) = \mathrm{Artin}_f(a_2)$.*

Proof. Suppose that $\mathrm{tp}_\sigma(a_1) = \mathrm{tp}_\sigma(a_2)$, and $f : q \to_G p$ is Galois. Then by saturation there is $\alpha \in \mathrm{Aut}(M, \sigma)$ where $\alpha(a_1) = a_2$. Let $b_1 \in f^{-1}\{a_1\}$, and say $\sigma(b_1) = g(b_1)$. Then $\mathrm{Artin}(a_1) = g^G$. But $\alpha(\sigma(b_1) = \alpha(g(b_1))$, and setting $b_2 := \alpha(b_1)$ we get $\sigma(b_2) = g(b_2)$, and $b_2 \in f^{-1}\{a_2\}$, so $\mathrm{Artin}_f(a_2) = g^G$.

For the converse, we shall define an L_σ-elementary bijection $\alpha : \mathrm{acl}_\sigma(a_1A) \to \mathrm{acl}_\sigma(a_2A)$. But first notice that by Remark 3.15 of [5], as $a_iA \subseteq F$, we have $\mathrm{acl}_\sigma(a_iA) = \mathrm{acl}^-(a_iA)$.

Now for any $b \in \mathrm{acl}^-(a_1A)$, witnessed minimally by $\theta(a_1, b)$ over A, we form $q(x, y) := p(x) \wedge \theta(x, y)$. By replacing $q(x, y)$ with $\mathrm{tp}_L(a_1, b_1, \ldots, b_n)/A$ where $(b_1, \ldots, b_n)$ is an enumeration of the realisations of $\theta(a_1, y)$, we may assume that projection $f : q \to_G p$ is Galois. Suppose $\sigma(b) = g_1(b)$, and pick $b' \in q(a_2, M)$, and say $g_2(b') = \sigma(b')$. Then as $\mathrm{Artin}_q(a_1) = \mathrm{Artin}_q(a_2)$ it follows that $g_2 = h^{-1}g_1h$ for some $h \in G$. Now define $c := hb'$, and for all $g \in G$ define $\alpha(g(b)) := g(c)$. Then for any $g \in G$, $\sigma(g(c)) = \sigma(g(hb')) = gh\sigma(b') = ghg_2(b') = ghg_2h^{-1}(c) = gg_1(c)$. Thus for all $g \in G$, $\alpha(\sigma(g(b))) = \alpha(g\sigma(b)) = \alpha(gg_1b_1) = gg_1c = \sigma(g(c))$. By the regular action of G it follows that $\alpha : q(a_1, M) \to q(a_2, M)$ is an L_σ-elementary bijection. Since b was originally chosen arbitrarily, by compactness it follows that there is an isomorphism of L_σ-structures $\alpha : \mathrm{acl}_\sigma(a_1A) \to \mathrm{acl}_\sigma(a_2A)$.

By Corollary 3.14 (i) of [5] it follows that $\mathrm{tp}_\sigma(a_1/A) = \mathrm{tp}_\sigma(a_2/A)$. ■

Corollary 4.3. *Let $D \subseteq F$ be an L_σ-definable set over $A \subseteq F$. Then $D = D' \cup E$ where D' is a finite union of Artin-definable sets over A, and $SU(E) < SU(D)$.*

Proof. Let $\{p_i : i \in I\}$ be the set of quantifier free $L_\sigma(A)$-types of maximal SU rank consistent with D. By Pillay's QE, see Lemma 3.17 of [5], this is the same as the set of consistent $L(A)$-types of maximal Morley rank, from which it follows that I is finite. Set $D_i := p_i(F) \cap D'$, and $D' := \bigcup_{i \in I} D_i$, and $E' := D \backslash D'$.

Proposition 4.2 above shows that there is a collection $\{X_{jk} : j \in J, k \in K\}$ of Artin-definable subsets of $p_i(F)$ over A, where $D_i = \bigcup_{j \in J} \bigcap_{k \in K} X_{jk}$. Then there are finite J_0, K_0 where $D = \bigcup_{j \in J_0} \bigcap_{k \in K_0} X_{jk}$ (if not then for every such J_0, K_0 the set $D_i \triangle (\bigcup_{j \in J_0} \bigcap_{k \in K_0} X_{jk})$ is consistent, which by compactness yields a contradiction). Thus taking common Galois covers, it follows that D_i is Artin-definable. ∎

4.1 Relationship to Pillay's QE

In Lemma 3.17 of [5], Pillay proves that any formula $\theta(x, a)$ in L_σ is equivalent to one of the form $\varphi(x, a) \wedge \exists y \psi(x, a, \overline{\sigma(y)})$ where

$$\overline{\sigma(y)} := (y, \sigma(y), \sigma^2(y), \ldots, \sigma^m(y))$$

for some m, and $\varphi(x, a)$ is the full partial L-type over a implied by $\theta(x, a)$, and $\psi(x, a, \bar{z})$ is such that $M \models \psi(x, a, \bar{z}) \to \bar{z} \in \mathrm{acl}(xa)$. We're interested in the case where $a \in F$ and $(M, \sigma) \models \theta(x, a) \to \sigma(x) = x$. We will also assume that $\mathrm{Md}(\varphi(x, a)) = \mathrm{Md}(\psi(x, a, \bar{w})) = 1$, as any L_σ-formula over F is equivalent to a disjunction of formulae of this form.

Let p and q be the generic types of $\varphi(x, a)$ and $\psi(x, a, \bar{z})$ respectively, and $f : q \to p$ be projection. By exchanging q (and ψ) with an (interdefinable) Galois cover, we may assume f is Galois with group G.

Now Pillay's QE specifies $\theta(x, a) \wedge p(x)$ by giving the **σ-form** $(\sigma^0, \sigma^1, \ldots, \sigma^m)$ on q: for $x \models p$, we have $M \models \theta(x, a) \leftrightarrow \exists y \psi(x, a, y, \sigma(y), \ldots, \sigma^m(y))$. But this is true if and only if there is $(x, a, y_1, y_2, \ldots, y_m) \in f^{-1}\{x\}$ where

$$\sigma(x, a, y_1, y_2, \ldots, y_m) = (x, a, y_2, y_3, \ldots, y_{m+1 (\mathrm{mod}\ n)})$$

which amounts to specifying the Artin class of x.

5 Measure

Definition 5.1. *Given $f : q \to_G p$ Galois, and $X := \{x \in p(F) : \mathrm{Artin}_q(x) \in S\}$ define*

$$\mathrm{Meas}(X) := \frac{|S|}{|G|}$$

We need to show that this is well-defined. For the moment we will refer to this quantity as $\mathrm{Meas}_f(X)$.

Lemma 5.2. *Suppose that $e : r \to_H p$ is Galois, and $X := \{x \in p(F) : \mathrm{Artin}_e(x) \in S$. suppose also $j : q \to r$ and , and $f = e \circ j$ is Galois. Then X can be Artin-defined via f, and $\mathrm{Meas}_e(X) = \mathrm{Meas}_f(X)$.*

Proof. Say f is Galois with group G. By Lemma 3.3, it is immediate that j is Galois with group $J := \{g \in G : j(y) = j(gy)\}$, and $G/J \cong H$ via the map $\nu : G \to H$ by $j(gy) = \nu(g)j(y)$.

Now we may suppose that S is a single conjugacy class of H: say $S = h_0^H$. Then $\nu^{-1}(S) = g_0^G J$, for some $g_0 \in \nu^{-1}\{h_0\}$. I claim that $X = \{x \in p(F) : \mathrm{Artin}_f(x) \in g_0^G J\}$. First suppose that $x \in p(F)$ and $\mathrm{Artin}_f(x) \in g_0^G J$. Then there is $z \in f^{-1}\{x\}$ where $\sigma(z) = g_0(z)$. Applying j we find that $j(\sigma(z)) = h_0 j(z)$ and so $\sigma(j(z)) = h_0 j(z)$, and thus $j(z)$ witnesses that $\mathrm{Artin}_e(x) \in h_0^H$.

Conversely suppose $x \in X$ and $y \in e^{-1}\{x\}$ where $\sigma(y) = h_0(y)$. Then pick $z \in j^{-1}\{y\}$. Then $\sigma(z) \in j^{-1}\{h_0 y\}$, and thus $\sigma(z) \in g_0 J j^{-1}\{y\}$, so $\sigma(z) = g_0 abz$ for some $a, b \in J$ and thus $\mathrm{Artin}_f(x) \in g_0^G J$.

Then $\mathrm{Meas}_f(X) = \frac{|g_0^G J|}{|G|} = \frac{|h_0^H| \cdot |J|}{|H| \cdot |J|} = \frac{|h_0^H|}{|H|} = \mathrm{Meas}_e(X)$. ■

Now we show that $\mathrm{Meas}(X)$ is well-defined.

Lemma 5.3. *Suppose that $f : q \to_G p$ and $e : r \to_J p$ are Galois and $X = \{x \in p(F) : \mathrm{Artin}_q(x) \in S\} = \{x \in p(F) : \mathrm{Artin}_r(x) \in T\}$ where $S \subseteq G$ and $T \subseteq J$. Then $\mathrm{Meas}_q(X) = \mathrm{Meas}_r(X)$.*

Proof. This straightforward. First we replace q by $\mathrm{tp}(f(y), y)$ for $y \models q$, and f by projection on to the forst co-ordinate, and similarly for e and r. Now we take a common Galois cover of q and r: let $s := \mathrm{tp}(x, f^{-1}\{x\}, e^{-1}\{x\})$, and define $\alpha : s \to q$ by projection onto the first two co-ordinates, $\beta : s \to r$ by projection onto the first and $(n+2)$nd co-ordinates (where $n = |f^{-1}\{x\}|$), and $\gamma : s \to p$ by projection onto the first co-ordinate. Then $\gamma = f \circ \alpha = e \circ \beta$ is Galois. Then by the previous lemma, X is definable via γ and $\mathrm{Meas}_f(X) = \mathrm{Meas}_\gamma(X) = \mathrm{Meas}_e(X)$. ■

6 Measure, finite additivity, and definable maps

We want to show that Meas is well-behaved under finite disjoint unions. The case where X and Y are given by different Artin-sets under the same Galois map is trivial: $\mathrm{Meas}(X \sqcup Y) = \frac{|S|+|T|}{|G|}$. To reduce to this case, given $f : q \to_G p$ and $e : r \to_J p$ defining X and Y respectively, we simply take a common Galois cover: $s := \mathrm{tp}(x, \overline{f^{-1}\{x\}}, \overline{e^{-1}\{x\}})$, and $g : s \to_G p$ by projection onto the first co-ordinate.

We now need to show that Meas is well-behaved under definable maps.

Lemma 6.1. *Suppose that q and p are complete L-types over $k \subset F$, and $\alpha : q \to p$ is an L-definable finite-to-1 map over k. Suppose X is an Artin-definable (over k) subset of $q(F)$, and $\alpha \restriction_X$ is l-to-1. Set $Y := \alpha(X)$. Then $l = \frac{\mathrm{Meas}(X)}{\mathrm{Meas}(Y)}$.*

Proof. Suppose X is defined via $\beta : s \to q$. Then define

$$r := \mathrm{tp}(y, \alpha^{-1}\{y\}, \beta^{-1}\{\alpha^{-1}\{y\}\})$$

Then, we replace q with $\mathrm{tp}(\alpha(x), x)$ for $x \models q$, and α with projection on to the first co-ordinate. Let $e : r \to q$ be projection on to the fist two co-ordinates, and $f : r \to p$ be projection on to the first co-ordinate. Then $f = \alpha \circ e$.

Moreover f is Galois with group G say, and as noted previously, setting $H := \{h \in G : e(z) = e(hz) \text{ for } z \in q\}$, we find that e is Galois with group H. Further X can be defined via e. Say $X = \{x \in q(F) : \mathrm{Artin}_e(x) \in S\}$. For the moment we will deal with the case that S is a single conjugacy class of H, say $S = g_0^H$.

Also we find that $Y = \{y \in p(F) : \mathrm{Artin}_f(y) \in g_0^G\}$: if $x \in X$ then there is $z \in e^{-1}\{x\}$ where $\sigma(z) = g_0(z)$, and thus there is $z \in f^{-1}\{\alpha(x)\}$ where $\sigma(z) = g_0(z)$, and thus $\mathrm{Artin}_f(\alpha(x)) \in g_0^G$; conversely if $\mathrm{Artin}_f(y) \in g_0^G$ then there is $z \in f^{-1}\{y\}$ where $\sigma(z) = g_0(z)$, and as $g_0 \in H$ we have $e(g_0 z) = e(z) = x$, say, so $x \in X$ and $y = \alpha(x)$, and so $y \in Y$.

Now set $A := \{a \models q : \sigma(a) = g_0(a)\}$. Then $e(A) = X$ since if $\sigma(a) = g_0(a)$ then $e(\sigma(a)) = e(g_0(a))$, and $\sigma(e(a)) = e(a)$, so $e(a) \in q(F)$. Also if $y \in e(A)$ then clearly $\mathrm{Artin}_e(y) \in g_0^H$.

Similarly $f(A) = Y$. Moroever $e \restriction_A$ has fibres of size $|C_H(g_0)|$: for $a, b \in A$, we have $e(a) = e(b)$ if and only if $b = g(a)$ for some $g \in H$. Also for $a, b \in A$ we have $g_0(a) = \sigma(a)$ and $g_0(b) = \sigma(b)$. So $g_0(g(a)) = \sigma(g(a)) = g(\sigma(a)) = g(g_0(a))$. Thus by regularity $gg_0 = g_0 g$, and $g \in C_H(g_0)$.

Similarly $f \restriction_A$ has fibres of size $|C_G(g_0)|$, and therefore $\alpha \restriction_X$ has fibres of size $\frac{|C_G(g_0)|}{|C_H(g_0)|} = \frac{|g_0^H|}{|H|} \cdot \frac{|G|}{|g_0^G|} = \frac{\mathrm{Meas}(X)}{\mathrm{Meas}(Y)}$ as required.

Now we turn to the case where S is a union of conjugacy classes of H. First we decompose Y into sets defined by single conjugacy classes of G. Say $Y = Y_1 \sqcup \ldots \sqcup Y_n$. Then decompose $X = X_1 \sqcup \ldots \sqcup X_n$, where $X_i = X \cap \alpha^{-1}(Y_i)$, and for each X_i we again decompose it into sets defined by a single conjugacy class of H: say $X_i = X_{i1} \sqcup \ldots \sqcup X_{im(i)}$. Then for each X_{ij} the above gives us that $\alpha \restriction_{X_{ij}}$ is $\frac{\mathrm{Meas}(X_{ij})}{\mathrm{Meas}(Y_i)}$-to-1. Hence $\alpha \restriction_{X_i}$ is $\frac{\sum_j \mathrm{Meas}(X_{ij})}{\mathrm{Meas}(Y_i)}$-to-1, that is to say $\frac{\mathrm{Meas}(X_i)}{\mathrm{Meas}(Y_i)}$-to-1. Moreover as the Y_i are disjoint it follows that $\mathrm{Meas}(X_i) = l \cdot \mathrm{Meas}(Y_i)$ for every i. Thus summing we find that $\frac{\sum_i \mathrm{Meas}(X_i)}{\sum_i \mathrm{Meas}(Y_i)} = l$, and so $\frac{\mathrm{Meas}(X)}{\mathrm{Meas}(Y)} = l$ as required. ∎

Lemma 6.2. *Suppose now that $\beta : p \to r$ is an L-definable map where the fibres have Morley degree 1. Suppose $f : q \to_G p$ is Galois, and $X = \{x \in p(F) : \mathrm{Artin}_f(x) \in S\}$. Then $\beta(X) = r(F)$ (hence this has measure 1) and the fibres of $\beta \restriction_X$ each have measure equal to* $\mathrm{Meas}(X)$.

Proof. Given $y \models r$, put $p_y := \alpha^{-1}\{y\}$, and $q_y := f^{-1}(p_y)$, and $f_y := f \restriction_{q_y}$. Then $f_y : q_y \to_G p_y$ is Galois, and if $y \in r(F)$ then the fibre of $\alpha \restriction_x$ above y is $X_y = \{x \in p_y(F) : \mathrm{Artin}_{f_y}(x) \in S\}$, which has measure $\frac{|S|}{|G|}$.

Furthermore given $y \in r(F)$, the fibre p_y has Morley degree 1 by assumption. Hence for any $g \in G$, it must contain a solution of $\sigma(x) = g(x)$ by the existential closure of SMA (see Lemma 3.8 of [5]). ■

Lemma 6.3. *Suppose now that $\gamma : p \to r$ is an L-definable map on stationary L-types all over $k \subseteq F$. Then there are q, $\alpha : p \to q$, and $\beta : q \to r$, where β is finite-to-1, α has fibres of Morley degree 1, and $\gamma = \beta \circ \alpha$.*

Proof. Given $x \models r$, define $p_x := \gamma^{-1}\{x\}$. Suppose that p_x^1 is a component of p_x over $acl_L(kx)$ of Morley degree 1. Let $c := Cb(p_x^1)$. By EI we may assume $c \in M^m$. Then let $q := \mathrm{tp}(c/k)$. Clearly $c \in acl(xk)$, and so the result follows. ■

Comment 6.4. For any definable map $\gamma : X \to p$ where $X \subseteq q(F)$ is Artin-definable, and where γ, p, q are defined over $k \subseteq F$, there is $\bar{\gamma} : q \to p$ where $\bar{\gamma} \restriction_X = \gamma$ since $q \vdash \exists y(\gamma(x) = y)$.

Lemma 6.5. *Suppose that q and p are complete L-types over $A \subset F$, and $\alpha : q \to p$ is an L-definable map over A. Then there are $\alpha : q \to r$ and $\beta : r \to p$, such that $\beta \circ \alpha = f$, and the fibres of α have Morley degree 1, and β is finite-to-1.*

Proof. Let c be the canonical base of a component of $f^{-1}\{x\}$, and set $r := \mathrm{tp}(c/A)$. By EI we may assume that $c \in M$. Then the result follows. ■

7 Extending measure to definable sets

We want to extend the measure to general definable sets. This will be routine using 4.3, but first we need the following lemma:

Lemma 7.1. *Suppose that p, q are complete stationary L-types over $A \subseteq F$, and $f : q \to_G p$ is Galois. Say $X = \{x \in p(F) |\ \mathrm{Artin}_q(x) \in S\}$.*

Let $F \supseteq B \supseteq A$, and let p', q' be the unique non-forking extensions of p and q respectively to B.

Then $f \restriction_{q'} : q' \to_G p'$ is Galois, and setting $X' := X \cap p'(F)$, we find that $X' = \{x \in p'(F) |\ \mathrm{Artin}_{q'} \in S\}$, and $\mathrm{Meas}(X') = \mathrm{Meas}(X)$.

Proof. Let $x \models p'$. Then $x \underset{A}{\overset{-}{\downarrow}} B$, say $U(x/B) = U(x/A) = n$. Also $x \models p$, so let $y \in f^{-1}\{x\}$. Then $U(xy/B) = U(x/By) + U(y/B) = U(y/Bx) + U(x/B)$. But x and y are interalgebraic over A (and hence over B), so $U(x/Bx) = U(y/Bx) = 0$. Thus $U(y/B) = n$ and similarly $U(y/A) = n$, so $y \underset{A}{\overset{-}{\downarrow}} B$, and thus $y \models q'$.

Hence $f^{-1}\{x\} \subseteq q'(M)$, and so G acts on $f \restriction_{q'}$ exactly as for f. The result follows. ■

Definition 7.2. *Let $D \subseteq F$ be an L-definable set over $a \subseteq F$ in the L-structure F. By Proposition 4.10 and Lemma 3.17 of [5], we can suppose D is L_σ-definable in (M, σ), say by $\varphi(x, a) \wedge \exists y \psi(x, a, \overline{\sigma(y)})$ where $\overline{\sigma(y)} := (y, \sigma(y), \sigma^2(y), \dots, \sigma^m(y))$ for some m, and $\varphi(x, a)$ is the full partial L-type over* *a implied by D, and* $M \models \psi(x, a, \bar{z}) \to \bar{z} \in \mathrm{acl}(xa)$*. Suppose* $\mathrm{Md}(\varphi(x, a)) = n$*, then we can split* $\varphi(x, a)$ *into* $\varphi_1(x, b), \ldots, \varphi_n(x, b)$ *over* $A := \mathrm{acl}(a)$*, each of degree 1.*

We can write $D_i := \varphi_i(M, b) \wedge \exists y \psi(x, a, \overline{\sigma(y)})$*. Then by 4.3* D_i *is (up to a set of lower rank) a finite union of Artin-definable sets. We define the measure of* D_i *and then D by addivity. This is well-defined by 7.1.*

It is straightforward to check that the behaviour under definable maps and disjoint unions carries across to this.

Lemma 7.3. *Given an L-formula* $\theta(x, y, a)$ *to be interpreted in the L-structure F, there are finitely many dimension/measure pairs* (d, μ) *which the family of sets* $\theta(F, b, a)$ *take as b varies in F.*

Proof. As in the definition above, we parse this in (M, σ) as a finite disjunction of sets of the form $\varphi(x, y, a) \wedge \exists z \psi(x, y, a, \overline{\sigma(z)})$, where $\mathrm{Md}(\varphi(x, y, a) = \mathrm{Md}(\psi(x, y, a, \bar{w})) = 1$. Bearing in mind Comment 4.1 above, it is enough to observe that for each such formula, as $(c, b, a, \bar{d}) \models \psi$ varies, $\mathrm{Mult}(\bar{d}/abc)$ is bounded (by compactness). Thus there is a bound on the size of the Galois groups, and hence there are only finitely many possible Artin classes, and thus measures, to consider. ■

Lemma 7.4. *Given an L-formula* $\theta(x, y, a)$ *to be interpreted in the L-structure F, for each dimension/measure pair* (d, μ) *there is an L-formula* $\delta_{(d,\mu)}(x, a)$ *which defines the set of b such that (Dim, Meas)*$(\theta(x, b, a)) = (d, \mu)$*.*

Proof. Again we consider θ as $\varphi(x, y, a) \wedge \exists z \psi(x, y, a, \overline{\sigma(z)})$. By the definability of types, we may define those b for which $\varphi(x, b, a)$ and $\psi(x, b, a, \bar{w})$ complete to specific stationary L-types in M $p(x)$ and $q(x, \bar{w})$ respectively. Then for each Galois map $f : q \to p$, by Lemma 7.3 above, there are only finitely many Artin-classes to consider, and the result follows from the relative definability of Artin-sets as remarked after 4.1 above. ■

8 Galois Maps and DMP

In this section we present a result of Hasson and Hrushovski, see Lemma 3.3 of [4]: loosely, a strongly minimal set has DMP if and only if for Galois maps, being automorphic is a definable property. We drop our standing assumptions, now $M \models T$ is strongly minimal with QE.

Given $f : q \to_G p$ Galois over a, and $b \models p$, let

$$A := \left\{ \alpha \restriction_{f^{-1}\{b\}} : \alpha \in \mathrm{Aut}(M/b) \right\}$$

Then of course $A \leq G^{\mathrm{op}}$.

Lemma 8.1. *If p is stationary, then* $\mathrm{Md}(q) = |G : A|$*.*

Proof. Say $\mathrm{Md}(q) = n$, and $q_1(y, ac), \ldots, q_n(y, ac)$ are the nonforking extensions. We define a map $(G^{\mathrm{op}} : A) \to \{1, \ldots, n\}$ by $gA \mapsto i$ where $g(q_1) = q_i$. Notice that this is well-defined: if $d, d' \models q_1$, then there is $a \in A$ where $a(d_1) = d_2$.

We show that this is a bijection. Pick $b \models p$ independent over c. Then, as p is stationary, $f^{-1}\{b\}$ must contain $d_i \models q_i$ for each i. Say $\beta_i \in G^{\mathrm{op}}$ is such that $\beta_i(d_1) = d_i$. Then $\beta_i \circ \beta_j^{-1}(d_j) = d_i$ and thus $\beta_i \circ \beta_j^{-1} \notin A$, and so $|G : A| \geq n$.

Conversely suppose $g^{-1}h \in A$. Then there is $a \in A$ such that $g^{-1}h(d_1) = a(d_1)$, and of course $a(d_1) \models q_1$ so $h(d_1) = g(a(d_1))$, and $h(q_1) = g(q_1)$. ■

Definition 8.2. A Galois map is *automorphic* if $A = G^{\mathrm{op}}$.

Theorem 8.3 (Hasson, Hrushovski, [4])**.** *T has the DMP if and only if being automorphic is definable, in the following sense:*

if $f_a : q_a \to_G p_a$ is an automorphic Galois map, then there is $\varphi \in \mathrm{tp}(a)$ such that whenever $a' \models \varphi$ then $f_{a'} : q_{a'} \to_G p_{a'}$ is automorphic too.

Proof. Suppose $\theta(\bar{x}, a)$ is a strongly minimal formula. Without loss of generality, we'll assume for generic $\bar{b} \models \theta$, that $\mathrm{MR}(b_1/a) = 1$ and $\bar{b} \in \mathrm{acl}(b_1 a)$. Then define $f : \theta(M^n, a) \to M$ by projection on to the first co-ordinate. Then pick $c \in M$ generic over a, and let $q_a := \mathrm{tp}(\overline{f^{-1}\{c\}}/a)$, and p_a be the generic 1-type in M over a. As q_a is interdefinable with the generic type of θ over a, it stationary over a. Thus $f_a : q_a \to_G p_a$ is automorphic.

If automorphicity is definable, then there is $\varphi \in \mathrm{tp}(a)$ which defines this. Thus for any $a' \models \varphi$, we have $q_{a'}$ is stationary over a', and so the generic type of $\theta(\bar{x}, a')$ is stationary over a', and thus $\theta(\bar{x}, a')$ is strongly minimal.

Conversely if automorphicity is not definable, then we assume that $f_a : q_a \to_G p_a$ is a counterexample to this. So for any $\varphi \in \mathrm{tp}(a)$ there exists $a' \models \theta$ such that $f_{a'} : q_{a'} \to_G p_{a'}$ is not automorphic, and thus $q_{a'}$ is not stationary, and so the generic type of $\theta(\bar{x}, a')$ is not stationary, and thus $\theta(\bar{x}, a')$ is not strongly minimal. ■

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