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ASYMPTOTIC CLASSES OF FINITE STRUCTURES

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§1. Introduction. In this paper we consider classes of finite structures where we have good control over the sizes of the definable sets. The motivating example is the class of finite fields: it was shown in [1] that for any formula $\phi(\bar{x}, \bar{y})$ in the language of rings, there are finitely many pairs $(d, \mu) \in \omega \times \mathbf{Q}^{>0}$ so that in any finite field $\boldsymbol{F}$ and for any $\bar{a} \in \mathbf{F}^m$ the size $|\phi(\mathbf{F}^n, \bar{a})|$ is "approximately" $\mu|\mathbf{F}|^d$. Essentially this is a generalisation of the classical Lang-Weil estimates from the category of varieties to that of the first-order-definable sets.

Motivated by this, we say that finite fields form a 1-*dimensional asymptotic class*. Macpherson and Steinhorn in [5] have studied these classes in abstract. Generalising this, in 2.1 below we define N-*dimensional asymptotic classes* for natural numbers $N \geq 1$, and begin to develop their general theory. In that definition, we have relaxed the asymptotic conditions (the meaning of "approximately" above), to encompass to the widest possible range of examples. We prove in corollary 2.8 that our classes lie within the general context of supersimple theories of finite rank.

In section 3 we consider how to define and interpret asymptotic classes inside one another, and in proposition 3.7 we show that the property of being an asymptotic class is invariant under bi-interpretations. In section 4 we give some examples of asymptotic classes, in particular, in proposition 4.1 we show that the smoothly approximable structures comprehensively studied in [2] fit into our framework. In section 5 we re-examine the relationship between dimension and D-rank. In section 6 we consider stable asymptotic classes. We show that stability can be detected within the finite structures in our context, and in proposition 6.5, we observe that stable asymptotic classes are locally modular.

NOTATION. If $\mathscr{U}$ is a (non-principal) ultrafilter on a set I and $\{M_i : I\}$ is a collection of $\mathscr{L}$-structures, we denote the ultraproduct by $P = \prod_{i \in I} M_i / \mathscr{U}$, and for $\bar{a} = (a_1, \ldots, a_n) \in P^n$, we shall write $\bar{a}(M) = (a_1(M), \ldots a_n(M))$ to mean the tuple of co-ordinates of $\bar{a}$ in M, so that for each $j \in \{1, \ldots, n\}$ we have $a_j = \prod_{i \in I} a_j(M_i)/\mathscr{U}$.

If M is an $\mathscr{L}$-structure, we write $\mathrm{Def}(M)$ for the collection of all parameter-definable sets in all powers of M.

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§2. Basic definitions and lemmas.

2.1. Definition. *Let $\mathscr{L}$ be a countable first order language, $N \in \omega$, and $\mathscr{C}$ a class of finite $\mathscr{L}$-structures. Then we say that $\mathscr{C}$ is a an N*-**dimensional asymptotic class** *if for every $\mathscr{L}$-formula $\phi(x, \bar{y})$ where $l(\bar{y}) = m$, there is a finite set of pairs $D \subseteq (\{0, \dots, N\} \times \mathbf{R}^{>0}) \cup \{(0, 0)\}$ and for each $(d, \mu) \in D$ a collection $\Phi_{(d,\mu)}$ of elements of the form $(M, \bar{a})$ where $M \in \mathscr{C}$ and $\bar{a} \in M^m$, so that $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ is a partition of $\{\{M\} \times M^m : M \in \mathscr{C}\}$, and*

$$|\phi(M, \bar{a})| - \mu|M|^{\frac{d}{N}} = o(|M|^{\frac{d}{N}})$$

as $|M| \longrightarrow \infty$ and $(M, \bar{a}) \in \Phi_{(d,\mu)}$.

Moreover each $\Phi_{(d,\mu)}$ is definable, that is to say $\{\bar{a} \in M^m : (M, \bar{a}) \in \Phi_{(d,\mu)}\}$ is uniformly $\emptyset$-definable across $\mathscr{C}$.

We may write D_ϕ for D, and will call $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ a **(definable) asymptotic partition**.

We write $h(\phi(M, \bar{a})) := (dim(\phi(M, \bar{a})), meas(\phi(M, \bar{a}))) := (d, \mu)$ where $(M, \bar{a}) \in \Phi_{(d,\mu)}$, except that if $d = \mu = 0$ we work with the convention that $dim(\phi(M, \bar{a})) = -1$.

We call $\mathscr{C}$ a **weak asymptotic class** *when $\mathscr{C}$ satisfies the asymptotic criteria, but the $\Phi_{(d,\mu)}$ may fail to be definable.*

Notice that an N-dimensional class is also an rN-dimensional class for all $r \in \omega$. We will usually choose N minimal so that the definition is satisfied.

It is immediate that the collection of all asymptotic classes is closed under taking subclasses and finite unions, and under expansions of the language by finitely many constants.

The above definition deals only with $\mathscr{L}$-formulae in one variable (plus parameters). However the next lemma shows that the corresponding statement for $\mathscr{L}$-formulae in more variables follows automatically.

2.2. Lemma. *If $\mathscr{C}$ is an N-dimensional asymptotic class, then for every $\mathscr{L}$-formula $\phi(\bar{x}, \bar{y})$ where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there is a finite set of pairs*

$$D \subseteq (\{0, \dots, Nn\} \times \boldsymbol{R}^{>0}) \cup \{(\boldsymbol{0}, \boldsymbol{0})\}$$

and a partition $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ of $\{\{M\} \times M^m : M \in \mathscr{C}\}$ so that

$$|\phi(M^n, \bar{a})| - \mu|M|^{\frac{d}{N}} = o(|M|^{\frac{d}{N}})$$

as $|M| \longrightarrow \infty$ and $(M, \bar{a}) \in \Phi_{(d,\mu)}$.

Again each $\Phi_{(d,\mu)}$ is definable.

Proof. We proceed by induction on n. The case $n = 1$ holds by assumption, so suppose that it holds for all $\mathscr{L}$-formulae $\phi_0(\bar{x}, \bar{y})$ where $l(\bar{x}) \leq n$. Consider now $\phi(z, \bar{x}, \bar{y})$ where $l(\bar{x}) = n$, $l(\bar{y}) = m$. By the case $n = 1$, there exists $\Delta = \{(e_1, \mu_1), \dots, (e_t, \mu_t)\} \subseteq \{0, \dots, N\} \times \mathbf{R}^{>0} \cup \{(0, 0)\}$ where $\{\Phi_{(e_i,\mu_i)} : 1 \leq i \leq t\}$ is a definable and asymptotic partition of $\{\{M\} \times M^{n+m} : M \in \mathscr{C}\}$. Say for $1 \leq i \leq t$, $M \in \mathscr{C}$, and $(\bar{b}, \bar{a}) \in M^{n+m}$ that $M \models \chi_i(\bar{b}, \bar{a}) \iff (M, \bar{b}, \bar{a}) \in \Phi_{(e_i,\mu_i)}$. Now by the inductive hypothesis, for each i there is $\Delta_i = \{(d_{i1}, v_{i1}), \dots, (d_{ir_i}, v_{ir_i})\} \subseteq \{0, \dots, Nn\} \times \mathbf{R}^{>0} \cup \{(0, 0)\}$ and a definable asymptotic partition $\{X_{ij} : 1 \leq j \leq r_i\}$ of $\{\{M\} \times M^m : M \in \mathscr{C}\}$ coresplonding to χ_i. Say $M \models \rho_{ij}(\bar{a}) \iff (M, \bar{a}) \in X_{ij} \iff h(\chi_i(M^n, \bar{a})) = (d_{ij}, v_{ij})$.

Given $M \in \mathscr{C}$ and $\bar{a} \in M^m$, there is a unique function $f\colon \{1,\dots,t\} \to \omega$ so that for all $i \in \{1,\dots,t\}$, $M \models \rho_{if(i)}(\bar{a})$. Notice that for each i, $f(i) \in \{1,\dots,r_i\}$, so the set of all possible such f is finite. We now fix f and consider M and $\bar{a}$ compatible with f.

Define $S_i(\bar{a}) := \{(z,\bar{x}) \in M^{n+1}\colon M \models \phi(z,\bar{x},\bar{a}) \wedge \chi_i(\bar{x},\bar{a})\}$. Then $\phi(M^{n+1},\bar{a}) = \bigsqcup_{i=1}^t S_i(\bar{a})$. We'll show that

$$|S_i(\bar{a})| - \mu_i \nu_{if(i)} |M|^{\frac{d_{if(i)}+e_i}{N}} = o\left(|M|^{\frac{d_{if(i)}+e_i}{N}}\right)$$

Suppose that $e_i > 0$ and $d_{if(i)} > 0$ (the cases where $e_i = 0$ or $d_{if(i)} = 0$ are similar, noting that if $d_{if(i)=0}$ then for large enough M, we have $|\chi(M^n,\bar{a})| = \nu_{if(i)}$, etc). Let $\varepsilon > 0$. Let $\varepsilon' := \mathrm{Min}\{\varepsilon, 3\mu_i\nu_{if(i)}\}$. Then for all sufficiently large M, we have

$$\left||\chi_i(M^n,\bar{a})| - \nu_{if(i)}|M|^{\frac{d_{if(i)}}{N}}\right| < \frac{\varepsilon'}{3\mu_i}|M|^{\frac{d_{if(i)}}{N}}$$

and for all $\bar{x} \in \chi_i(M^n,\bar{a})$,

$$\left||\phi(M,\bar{x},\bar{a})| - \mu_i|M|^{\frac{e_i}{N}}\right| < \frac{\varepsilon'}{3\nu_{if(i)}}|M|^{\frac{e_i}{N}}$$

As

$$|S_i(\bar{a})| = \sum_{\bar{x}\in\chi_i(M^n,\bar{a})} |\phi(M,\bar{x},\bar{a})|$$

we have

$$\left(\nu_{if(i)} - \frac{\varepsilon'}{3\mu_i}\right)\left(\mu_i - \frac{\varepsilon'}{3\nu_{if(i)}}\right)|M|^{\frac{d_{if(i)}+e_i}{N}} < |S_i(\bar{a})|$$
$$< \left(\nu_{if(i)} + \frac{\varepsilon'}{3\mu_i}\right)\left(\mu_i + \frac{\varepsilon'}{3\nu_{if(i)}}\right)|M|^{\frac{d_{if(i)}+e_i}{N}}$$

and so

$$\left||S_i(\bar{a})| - \mu_i\nu_{if(i)}|M|^{\frac{d_{if(i)}+e_i}{N}}\right| < \varepsilon|M|^{\frac{d_{if(i)}+e_i}{N}} s$$

as required.

Now let $d := \mathrm{Max}\{d_{if(i)}+e_i\colon i \in \{1,\dots,t\}\}$, $A := \{i \in \{1,\dots,t\}\colon d_{if(i)}+e_i = d\}$, and $\mu := \sum_{i\in A}\mu_i\nu_{if(i)}$. Then $\sum_{i\in A}|S_i(\bar{a})| - \mu|M|^{\frac{d}{N}} = o(|M|^{\frac{d}{N}})$. But for each $i \notin A$ we have $d_{if(i)} + e_i < d$, so in fact $\sum_{i=1}^t |S_i(\bar{a})| - \mu|M|^{\frac{d}{N}} = o(|M|^{\frac{d}{N}})$, that is $|\phi(M^{n+1},\bar{a})| - \mu|M|^{\frac{d}{N}} = o(|M|^{\frac{d}{N}})$.

As the set of possible f earlier was finite, it follows that the corresponding set of (d,μ) is also finite. Notice also that each $d_{if(i)} \in \{0,\dots,Nn\}$, and $e_i \in \{0,\dots,N\}$, so $d \in \{0,\dots,N(n+1)\}$.

Now we need to show definability. Consider some specific (d,μ), and let $\{f_1,\dots,f_p\}$ be the set of functions f which yield (d,μ). Define

$$\zeta_{(d,\mu)}(\bar{y}) := \bigvee_{j=1}^{p}\bigwedge_{i=1}^{t}\rho_{if_j(i)}(\bar{y})s$$

This defines

$$\bigcup_{j=1}^{p}\bigcap_{i=1}^{t} X_{i f_j(i)}$$

which is the desired set. ■

We shall want to speak of dimension and measure in the context of infinite ultraproducts of members of an asymptotic class:

2.3. DEFINITION. *Let $\mathscr{C}$ be an N-dimensional asymptotic class in a language $\mathscr{L}$, P be an infinite ultraproduct of members of $\mathscr{C}$, and $\phi(P^n, \bar{a})$ be a $\mathscr{L}$-definable set. Then as D_ϕ is finite, for some unique $(d, \mu) \in D_\phi$ we have*

$$\{M \in \mathscr{C} : (M, \bar{a}(M)) \in \Phi_{(d,\mu)}\} \in \mathscr{U}.$$

In this sense we write $(dim, meas)(\phi(P^n, \bar{a})) := (d, \mu)$.

2.4. DEFINITION. *Let $\mathscr{C}$ be a class of finite $\mathscr{L}$-structures. Then for each $Q \in \omega$ we define $\mathscr{C}^{\geq Q} := \{M \in \mathscr{C} : |M| \geq Q\}$, and*

$$Th(\mathscr{C}) := \{\sigma : \sigma \text{ is an } \mathscr{L}\text{-sentence and } \exists Q \in \omega \text{ such that } \forall M \in \mathscr{C}^{\geq Q} \text{ we have } M \models \sigma\}$$

2.5. PROPOSITION. *Let $\mathscr{C}$ be an asymptotic class. Then the following are equivalent*:

1. *For each $\phi(\bar{x}) \in \mathscr{L}$ there are $(d, \mu) \in D_\phi$ and $Q \in \omega$ such that for all $M \in \mathscr{C}^{\geq Q}$ we have $h(\phi(M^n)) = (d, \mu)$.*
2. *$Th(\mathscr{C})$ is complete.*
3. *For any infinite ultraproducts P_1 and P_2 of members of $\mathscr{C}$ we have that $P_1 \equiv P_2$.*

PROOF. The fact that (2) and (3) are equivalent is straightforward and holds for any class of finite $\mathscr{L}$-structures.

Suppose now that (1) holds. Let P_1 and P_2 be infinite ultraproducts of members of $\mathscr{C}$, and let σ be an $\mathscr{L}$-sentence. Then for $i \in \{1, 2\}$ we have

$$P_i \models \sigma \Leftrightarrow h(\{x \in P_i : (x = x) \wedge \sigma\}) = h(P_i)$$

But by (1) we know that there is $Q \in \omega$ where $h(\{x \in M : (x = x) \wedge \sigma\})$ is constant across $\mathscr{C}^{\geq Q}$. So in fact

$$\begin{aligned} P_1 \models \sigma &\Leftrightarrow \text{for } M \in \mathscr{C}^{\geq Q} \text{ we have } h(\{x \in M : (x = x) \wedge \sigma\}) = h(M) \\ &\Leftrightarrow P_2 \models \sigma \end{aligned}$$

Now suppose that (1) fails. Then there are $\phi(\bar{x}) \in \mathscr{L}$ and arbitrarily large pairs $M_1, M_2 \in \mathscr{C}$ such that $h(\phi(M_1^n)) \neq h(\phi(M_2^n))$. As D_ϕ is finite, we may find $(d_1, \mu_1), (d_2, \mu_2) \in D_\phi$ and unbounded sequences $\{M_{1j} : j \in \omega\}, \{M_{2j} : j \in \omega\} \subset \mathscr{C}$ so that for all $j \in \omega$ we have

$$h(\phi(M_{1j})) = (d_1, \mu_1) \neq (d_2, \mu_2) = h(\phi(M_{2j}))$$

Now let $\mathscr{U}$ be a non-principal ultrafilter on ω and for $i \in \{1, 2\}$ let $P_i := \prod_{j\in\omega} M_{ij}/\mathscr{U}$. By the definability of dimension/measure in $\mathscr{C}$ there are sentences $\sigma_1, \sigma_2 \in \mathscr{L}$ where for $i \in \{1, 2\}$ and all $M \in \mathscr{C}$

$$M \models \sigma_i \Leftrightarrow h(\phi(M^n)) = (d_i, \mu_i)$$

Hence $P_1 \not\equiv P_2$ as $P_1 \models \sigma_1 \wedge (\neg\sigma_2)$ but $P_2 \models (\neg\sigma_1) \wedge \sigma_2$. ■

For the reader's convenience, we recall the definition of Shelah's D-rank (see for instance 5.1.13 of [9]):

2.6. Definition. *Let M be a first order structure. We define the D-**rank** of a formula $\phi(\bar{x}, \bar{a})$ recursively as follows*:

- $D(\phi(\bar{x}, \bar{a})) \geq 0$ *if* $\phi(\bar{x}, \bar{a})$ *is consistent.*
- *For all ordinals* α, $D(\phi(\bar{x}, \bar{a})) \geq \alpha + 1$ *if there exist* $\psi(\bar{x}, \bar{y})$ *and a sequence* $\{\bar{c}_i : i \in \omega\}$, *indiscernible over* $\bar{a}$, *so that*:
  - *For each* $i \in \omega$, $M \models \psi(\bar{x}, \bar{c}_i) \rightarrow \phi(\bar{x}, \bar{a})$
  - *For each* $i \in \omega$, $D(\psi(\bar{x}, \bar{c}_i)) \geq \alpha$
  - *There exists* $k \in \omega$, *so that* $\{\psi(\bar{x}, \bar{c}_i) : i \in \omega\}$ *is* k*-inconsistent.*
- *For limit ordinals* α, $D(\phi(\bar{x}, \bar{a})) \geq \alpha$, *if for all* $\beta < \alpha$, $D(\phi(\bar{x}, \bar{a})) \geq \beta$.

Recall that a first-order theory T is supersimple if and only if every formula in every model of T has ordinal D-rank.

We now prove a result linking the dimension and D-rank in ultraproducts of asymptotic classes. In forthcoming work in [4] we tackle the same question at the more general level of infinite measurable structures (see 2.9 below). However we retain the current proof here for its finitary nature.

2.7. Proposition. *Let $\mathscr{C}$ be an N-dimensional asymptotic class, and $P = \prod M/\mathscr{U}$ an infinite ultraproduct of members of $\mathscr{C}$. Then for all $\mathscr{L}$-formulae $\phi(\bar{x}, \bar{y})$ and all $\bar{a} \in P^m$, $D(\phi(P^n, \bar{a})) \leq \dim(\phi(P^n, \bar{a}))$.*

Proof. By shrinking $\mathscr{C}$ if necessary, we may assume that $\mathscr{U}$ is a non-principal ultrafilter on $\mathscr{C}$. We proceed by induction, and show that if $D(\phi(P^n, \bar{a}) \geq r$, then also $\dim(\phi(P^n, \bar{a}) \geq r$. Notice that the case $r = 0$ is trivial as it is just the condition that $\phi(P^n, \bar{a})$ is non-empty, and $r = 1$ is the condition that $\mathscr{U}$ contains a set in which $\phi(M^n, \bar{a}(M))$ is unbounded. Suppose now that the result holds for all $r \leq s$, and that $D(\phi(P^n, \bar{a})) \geq s + 1$. Then there exist $\psi(\bar{x}, \bar{z})$ and an indiscernible sequence $\{\bar{c}_i : i \in \omega\}$ in P^m such that:

- For all $i \in \omega$, $P \models \psi(\bar{x}, \bar{c}_i) \rightarrow \phi(\bar{x}, \bar{a})$
- For all $i \in \omega$, $D(\psi(\bar{x}, \bar{c}_i)) \geq s$
- There exists $k \in \omega$ so that $\{\psi(\bar{x}, \bar{c}_i) : i \in \omega\}$ is k-inconsistent.

Define

$$U_i' := \{M \in \mathscr{C} : \psi(M^n, \bar{c}_i(M)) \subseteq \phi(M^n, \bar{a}(M))\},$$
$$V_i' := \{M \in \mathscr{C} : \dim(\psi(M^n, \bar{c}_i(M))) \geq s\},$$

and for any distinct $i_1, \ldots, i_k \in \omega$ define

$$W_{i_1,\ldots,i_k} := \{M \in \mathscr{C} : \bigwedge_{j=1}^{k} \psi(M^n, \bar{c}_{i_j}(M)) = \emptyset\}.$$

Then by the inductive hypothesis we know that $U_i', V_i', W_{i_1,\ldots,i_k} \in \mathscr{U}$. Also by the inductive hypothesis we know that

$$\{M \in \mathscr{C} : \dim(\phi(M^n, \bar{a}(M)) \geq s\} \in \mathscr{U}.$$

Hence either

$$V := \{M \in \mathscr{C} : \dim(\phi(M^n, \bar{a}(M)) = s\} \in \mathscr{U},$$

or

$$\{M \in \mathscr{C} : \dim(\phi(M^n, \bar{a}(M)) \geq s+1\} \in \mathscr{U}.$$

Suppose for a contradiction that the former holds. For each $Q \geq k$ define

$$V_Q := \Big(\bigcap_{i=0}^{Q} U_i'\Big) \cap \Big(\bigcap_{i=0}^{Q} V_i'\Big) \cap \Big(\bigcap_{0\leq i_1<i_2<\cdots<i_k}^{i_k=Q} W_{i_1,\ldots,i_k}\Big) \cap V$$

Notice that $V_Q \in \mathscr{U}$ and that $V_{Q+1} \subseteq V_Q$. Also for convenience we write $A_i := \psi(M^n, \bar{c}_i(M))$ (which M we're working in will be clear from context). Now for all $M \in V_Q$ and all $i \leq Q$, the following hold: $A_i \subseteq \phi(M^n, \bar{a}(M))$, $\dim(A_i) \geq s$, and $\dim(\phi(M^n, \bar{a}(M))) = s$. Hence for all sufficiently large $M \in V_Q$, it must hold that $\dim(A_i) = s$, and (by shrinking V_Q if necessary) we may suppose that this holds for all $M \in V_Q$. Now,

$$\begin{aligned} |\phi(M^n, \bar{a}(M))| &\geq |\bigcup_{i=0}^{Q} A_i| \\ &= \sum_{i=0}^{Q} |A_i| - \sum_{0\leq i_1<i_2}^{i_2=Q} |A_{i_1} \cap A_{i_2}| + \ldots \\ &\quad + (-1)^k \sum_{0\leq i_1<\cdots<i_{k-1}}^{i_{k-1}=Q} |A_{i_1} \cap \cdots \cap A_{i_{k-1}}| \end{aligned}$$

Moreover, by indiscernibility, for all $j, i_1, \ldots, i_{j+1}, i_1', \ldots, i_{j+1}' \leq Q$, we have $h(\psi(P^n, \bar{c}_{i_1}) \cap \cdots \cap \psi(P^n, \bar{c}_{i_{j+1}})) = h(\psi(P^n, \bar{c}_{i_1}') \cap \cdots \cap \psi(P^n, \bar{c}_{i_{j+1}}'))$. Thus we may find $\widetilde{V_Q} \subseteq V_Q$ in $\mathscr{U}$, $\mu \in \boldsymbol{R}^{>0}$, and $(d_0, \mu_0), \ldots, (d_{k-2}, \mu_{k-2})$, so that for all $M \in \widetilde{V_Q}$, we have $h(\phi(M^n, \bar{a}(M))) = (s, \mu)$, and for all $j \in \{0, \ldots, k-2\}$ and all $i_1, \ldots, i_{j+1} \in \omega$, we have $h(A_{i_1} \cap \cdots \cap A_{i_{j+1}}) = (d_j, \mu_j)$.

Thus for all $\varepsilon > 0$ and all sufficiently large $M \in \widetilde{V_Q}$,

$$\begin{aligned} |\phi(M^n, \bar{a}(M))| \geq (Q+1)(\mu_0 - \varepsilon)|M|^{\frac{d_0}{N}} - \frac{(Q+1)Q}{2}(\mu_1 + \varepsilon)|M|^{\frac{d_1}{N}} + \ldots \\ + (-1)^k \frac{(Q+1)Q\ldots(Q+2-k)}{(k-1)!} \Big(\mu_{k-2} + (-1)^{k-1}\varepsilon\Big) |M|^{\frac{d_{k-2}}{N}} \end{aligned}$$

Claim.

$$s = d_0 > d_1 > \cdots > d_{k-2} > -1$$

Proof of Claim. We know that $s = d_0 \geq d_1 \geq \cdots \geq d_{k-2} \geq -1$. Suppose for a contradiction that strict inequalities do not hold throughout. Let $l := \mathrm{Max}\{l' : d_{l'} = d_{l'+1}\}$. Then for all $\varepsilon > 0$ and all sufficiently large $M \in \widetilde{V_Q}$,

$$(\mu_l + \varepsilon)|M|^{\frac{d_l}{N}} \geq |A_0 \cap \cdots \cap A_l| \geq |\bigcup_{i=l+1}^{Q} A_0 \cap \cdots \cap A_l \cap A_i|$$

Now using the inclusion/exclusion principle we get

$$
\begin{aligned}
(\mu_l+\varepsilon)|M|^{\frac{d_l}{N}} \geq & \sum_{i=l+1}^{Q} |A_0\cap\cdots\cap A_l\cap A_i| \\
& - \sum_{l+1\leq i_1<i_2}^{i_2=Q} |A_0\cap\cdots\cap A_l\cap A_{i_1}\cap A_{i_2}| + \dots \\
& + (-1)^{k-1-l} \sum_{l+1\leq i_1<\dots i_{k-2-l}}^{i_{k-2-l}=Q} |A_0\cap\cdots\cap A_l\cap A_{i_1}\cap\cdots\cap A_{i_{k-2-l}}|
\end{aligned}
$$

So

$$
\begin{aligned}
(\mu_l+\varepsilon)|M|^{\frac{d_l}{N}} \geq (Q-l)(\mu_{l+1}-\varepsilon)|M|^{\frac{d_l}{N}} - \binom{Q-l}{2}(\mu_{l+2}+\varepsilon)|M|^{\frac{d_{l+2}}{N}} + \dots \\
+ (-1)^{k-1-l}\binom{Q-l}{k-l-2}(\mu_{k-2}+(-1)^{k-2-l}\varepsilon)|M|^{\frac{d_{k-2}}{N}} \qquad (1)
\end{aligned}
$$

However for large enough Q and small enough ε, we have $(Q-l)(\mu_{l+1}-\varepsilon) > \mu_l+\varepsilon$, and then by taking large enough $M \in \widetilde{V_Q}$ we may contradict equation (1), as $d_l > d_{l+2} > \cdots > d_{k-2}$. QED Claim

A similar argument now completes the proof: we have, for all $Q \in \omega$, all $\varepsilon > 0$ and all sufficiently large $M \in \widetilde{V_Q}$

$$
\begin{aligned}
(\mu+\varepsilon)|M|^{\frac{s}{N}} & \geq |\phi(M^n,\bar{a}(M))| \\
& \geq (Q+1)(\mu_0-\varepsilon)|M|^{\frac{s}{N}} - \binom{Q+1}{2}(\mu_1+\varepsilon)|M|^{\frac{d_1}{N}} + \dots \\
& \quad + (-1)^k\binom{Q+1}{k-1}\left(\mu_{k-2}+(-1)^{k-1}\varepsilon\right)|M|^{\frac{d_{k-2}}{N}}
\end{aligned}
$$

where $s > d_1 > \cdots > d_{k-2}$, so just as in the proof of the claim, for large enough Q, small enough ε, and large enough $M \in \widetilde{V_Q}$, we have a contradiction. ■

2.8. Corollary. *Any ultraproduct of an N-dimensional asymptotic class is supersimple of D-rank at most N.*

Proof. Simply apply the previous proposition to $x = x$. ■

There is a broader class of infinite structures which admit dimension and measure, which have been studied in [5]. We give the definition here:

2.9. Definition. *An infinite $\mathscr{L}$-structure M is* **measurable** *if there is a function $h\colon \operatorname{Def}(M) \to (\omega\times\mathbf{R}^{>0})\cup\{(0,0)\}$ (we write $h(X) = (\dim(X), \operatorname{meas}(X))$ such that the following hold*:

1. *For each $\mathscr{L}$-formula $\phi(\bar{x},\bar{y})$ there is a finite set $D \subset (\omega\times\mathbf{R}^{>0})\cup\{(0,0)\}$, so that for all $\bar{a}\in M^m$ we have $h(\phi(M^n,\bar{a}))\in D$.*
2. *If $\phi(M^n,\bar{a})$ is finite then $h(\phi(M^n,\bar{a})) = (0, |\phi(M^n,\bar{a})|)$.*
3. *If $\phi(M^n,\bar{a})$ is a definable set and $\psi(\bar{x},\bar{y})$ and $\{\bar{c}_i : i\in\omega\}$ are indiscernible over $\bar{a}$ such that $M \models \phi(\bar{x},\bar{c}_i)\to\phi(\bar{x},\bar{a})$, $\dim(\phi(\bar{x},\bar{c}_i)) \geq n$, and for some k the collection $\{\phi(\bar{x},\bar{c}_i) : i\in\omega\}$ is k-inconsistent, then $\dim(X)\geq n+1$.*

4. *Let $X, Y \in \mathrm{Def}(M)$ and $f : X \to Y$ be a definable surjection. Then there is $r \in \omega$ and $(d_1, \mu_1), \dots, (d_r, \mu_r) \in (\omega \times \mathbf{R}^{>0}) \cup \{(0,0)\}$ so that if*

$$Y_i := \{\bar{y} \in Y : h(f^{-1}(\bar{y})) = (d_i, \mu_i)\},$$

then $Y = Y_1 \cup \cdots \cup Y_r$ is a partition of Y into non-empty disjoint definable sets. Let $h(Y_i) = (e_i, \nu_i)$ for $i \in \{1, \dots, r\}$. Also let $c := \mathrm{Max}\{d_1 + e_1, \dots, d_r + e_r\}$, and suppose this maximum is attained by $d_1 + e_1, \dots, d_s + e_s$. Then $h(X) = (c, \mu_1 \nu_1 + \cdots + \mu_s \nu_s)$.
5. *For every $\mathscr{L}$-formula $\phi(\bar{x}, \bar{y})$ and all $(d, \mu) \in D_\phi$, the set*

$$\{\bar{a} \in M^m : h(\phi(M^n, \bar{a})) = (d, \mu)\}$$

is $\emptyset$-definable.

If $X \in \mathrm{Def}(M)$ and $h(X) = (d, \mu)$, we call d the **dimension** *of X and μ the* **measure** *of X, and h the* **measuring function**.

We say that a complete theory T is **measurable** *if it has a measurable model.*

Notice that 3 above ensures that for all measurable structures M and all $X \in \mathrm{Def}(M)$ we have $D(X) \leq \dim(X)$. In forthcoming work (see [4]) it is shown that this condition in fact follows from the others.

§3. New classes from old. Consider now a weak asymptotic class $\mathscr{C}$ in a language $\mathscr{L}$, and a uniformly definable set X_M for each $M \in \mathscr{C}$. I want to treat $\{X_M : M \in \mathscr{C}\}$ as a class of finite structures. First we have to consider which first-order languages are appropriate for this class. There are many languages one might choose subject to the context, but the following gives a minimum criterion that any of them should satisfy.

3.1. Definition. *Let $\mathscr{C}$ be a weak asymptotic class $\mathscr{C}$ in a language $\mathscr{L}$, and let $\phi(\bar{x}, \bar{y})$ be an $\mathscr{L}$-formula with $l(\bar{x}) = n$ and $l(\bar{y}) = m$. Define*

$$\phi(\mathscr{C}) := \{\phi(M^n, \bar{a}) : M \in \mathscr{C}, \bar{a} \in M^m\}$$

Let $\mathscr{L}'$ be a first-order language such that $\phi(\mathscr{C})$ is a class of $\mathscr{L}'$-structures. We say $\mathscr{L}'$ is **suitable** *if for any $\psi'(\bar{z}', \bar{w}') \in \mathscr{L}'$ (say $l(\bar{z}') = s$ and $l(\bar{w}') = r'$) there is $\psi(\bar{z}, \bar{w}) \in \mathscr{L}$ (where $l(\bar{z}) = n \cdot s$ and $l(\bar{w}) = r$) so that for every $X = \phi(M^n, \bar{a}) \in \phi(\mathscr{C})$ and every $\bar{b}' \in X^{r'}$ there is $\bar{b} \in M^r$ so that $\psi'(X^s, \bar{b}') = \psi(M^{n \cdot s}, \bar{b})$.*

3.2. Lemma. *If $\mathscr{C}$ is an N-dimensional weak asymptotic class, and $\phi(\bar{x}, \bar{y})$ is an $\mathscr{L}$-formula with $l(\bar{x}) = n$ and $l(\bar{y}) = m$. Then $\phi(\mathscr{C})$ is a weak asymptotic class in any suitable language $\mathscr{L}'$.*

Proof. For $(d, \mu) \in D_\phi$ define $\phi(\mathscr{C})_{(d,\mu)} := \{\phi(M^n, \bar{a}) : (M, \bar{a}) \in \Phi_{(d,\mu)}\}$. As D_ϕ is finite, it is sufficient to show that $\phi(\mathscr{C})_{(d,\mu)}$ is a d-dimensional weak asymptotic class in $\mathscr{L}'$, under the function $H(\psi(\phi(M^n, \bar{a}), \bar{b})) := (e, \frac{\nu}{\mu^{\frac{e}{d}}})$, where $h(\psi(M^n, \bar{b}) \cap \phi(M^n, \bar{a})) = (e, \nu))$. For convenience we will suppress the parameters $\bar{a}$ occuring in ϕ.

Let $\varepsilon > 0$ and choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that $\varepsilon_1 < \frac{\mu}{2}$, $\left|\frac{v \pm \varepsilon_2}{(\mu-\varepsilon_1)^{\frac{e}{d}}} - \frac{v}{\mu^{\frac{e}{d}}}\right| < \varepsilon$, and $\left|\frac{(v \pm \varepsilon_2)(1-\frac{2\varepsilon_1}{\mu})^{\frac{e}{d}}}{(\mu-\varepsilon_1)^{\frac{e}{d}}} - \frac{v}{\mu^{\frac{e}{d}}}\right| < \varepsilon$. Then for large enough $M \in \mathscr{C}$ we have

$$\left||\phi(M^n)| - \mu|M|^{\frac{d}{N}}\right| < \varepsilon_1 |M|^{\frac{d}{N}} \tag{2}$$

and

$$\left||(\phi \wedge \psi)(M^n)| - v|M|^{\frac{e}{N}}\right| < \varepsilon_2 |M|^{\frac{e}{N}} \tag{3}$$

Clearly (2) gives that

$$|M|^{\frac{d}{N}} < \frac{1}{\mu}\left(|\phi(M^n)| + \varepsilon_1|M|^{\frac{d}{N}}\right)$$

and so using (2) again

$$|M|^{\frac{d}{N}} < \frac{1}{\mu}\left(|\phi(M^n)| + \varepsilon_1\left(\frac{1}{\mu}\left(|\phi(M^n)| + \varepsilon_1|M|^{\frac{d}{N}}\right)\right)\right)$$

Continuing in a similar way, we find that for all $r \in \omega$ we have

$$|M|^{\frac{d}{N}} < \frac{1}{\mu}\left(1 + \frac{\varepsilon_1}{\mu} + \cdots + \frac{{\varepsilon_1}^r}{\mu^r}\right)|\phi(M^n)| + \frac{{\varepsilon_1}^{r+1}}{\mu^{r+1}}|M|^{\frac{d}{N}}$$

and so taking the limit as $r \to \infty$ we get that

$$|M|^{\frac{d}{N}} < \frac{1}{\mu - \varepsilon_1}|\phi(M^n)| \tag{4}$$

and substituting this back into (2), we get

$$\mu|M|^{\frac{d}{N}} > |\phi(M^n)| - \varepsilon_1|M|^{\frac{d}{N}} > \left(1 - \frac{\varepsilon_1}{\mu - \varepsilon_1}\right)|M|^{\frac{d}{N}}$$

so by this and (4) we get

$$\frac{1 - \frac{2\varepsilon_1}{\mu}}{\mu - \varepsilon_1}|\phi(M^n)| < |M|^{\frac{d}{N}} < \frac{1}{\mu - \varepsilon_1}|\phi(M^n)|$$

and therefore

$$\frac{\left(1 - \frac{2\varepsilon_1}{\mu}\right)^{\frac{e}{d}}}{(\mu - \varepsilon_1)^{\frac{e}{d}}}|\phi(M^n)|^{\frac{e}{d}} < |M|^{\frac{e}{N}} < \frac{1}{(\mu - \varepsilon_1)^{\frac{e}{d}}}|\phi(M^n)|^{\frac{e}{d}}$$

Thus using (3)

$$\frac{(v - \varepsilon_2)\cdot\left(1 - \frac{2\varepsilon_1}{\mu}\right)^{\frac{e}{d}}}{(\mu - \varepsilon_1)^{\frac{e}{d}}}|\phi(M^n)|^{\frac{e}{d}} < |(\phi \wedge \psi)(M^n)| < \frac{v + \varepsilon_2}{(\mu - \varepsilon_1)^{\frac{e}{d}}}|\phi(M^n)|^{\frac{e}{d}}$$

and by choice of ε_1 and ε_2,

$$\left(\frac{v}{\mu^{\frac{e}{d}}} - \varepsilon\right)|\phi(M^n)|^{\frac{e}{d}} < |(\phi \wedge \psi)(M^n)| < \left(\frac{v}{\mu^{\frac{e}{d}}} + \varepsilon\right)|\phi(M^n)|^{\frac{e}{d}}$$

as required. ■

Now we look at the definable quotients of asymptotic classes. Again, we first have to consider the question of which languages to use:

3.3. Definition. *Let $\mathscr{C}$ be an N-dimensional asymptotic class, and $E(\bar{x}_1, \bar{x}_2, \bar{y})$ an $\mathscr{L}$-formula with $l(\bar{x}_1) = l(\bar{x}_2) = n$, $l(\bar{y}) = m$.*

$$\mathscr{C}/E := \{M^n/E_{\bar{a}} \colon M \in \mathscr{C} \text{ and } E(\bar{x}_1, \bar{x}_2, \bar{a}) \text{ defines an equivalence relation } E_{\bar{a}} \text{ on } M^n\}$$

Let $\mathscr{L}'$ be a first-order language such that $\mathscr{C}/E$ is a class of $\mathscr{L}'$-structures. We say that $\mathscr{L}'$ is **suitable** *if for every $\psi'(\bar{z}', \bar{w}') \in \mathscr{L}'$ (say $l(\bar{z}') = s$ and $l(\bar{w}') = r'$) there is $\psi(\bar{z}, \bar{w}) \in \mathscr{L}$ (where $l(\bar{z}) = n \cdot s$ and $l(\bar{w}) = r$) so that for every $X = M^n/E_{\bar{a}} \in \mathscr{C}/E$ and every $\bar{b}' \in X^{r'}$ there is $\bar{b} \in M^r$ so that $\psi'(X^s, \bar{b}') = \psi(M^{n \cdot s}, \bar{b})/E_{\bar{a}}$.*

Also define $\mathscr{L}'_E$ to be the disjoint union of $\mathscr{L}$ and $\mathscr{L}'$, along with the natural projection map. We form a class of $\mathscr{L}'_E$-structures in the obvious way:

$$\mathscr{C}_E := \{(M, \bar{a}) \cup (M^n/E_{\bar{a}}) \colon E(\bar{x}_1, \bar{x}_2, \bar{a}) \text{ defines an equivalence relation } E_{\bar{a}} \text{ on } M^n\}$$

3.4. Lemma. *Let $\mathscr{C}$ be an N-dimensional asymptotic class, and $E(\bar{x}_1, \bar{x}_2, \bar{y})$ an $\mathscr{L}$-formula with $l(\bar{x}_1) = l(\bar{x}_2) = n$, $l(\bar{y}) = m$. Then $\mathscr{C}/E$ is a weak Nn-dimensional asymptotic class in any suitable language $\mathscr{L}'$, and $\mathscr{C}_E$ is an N-dimensional asymptotic class in the corresponding language $\mathscr{L}'_E$.*

Proof. We consider $M \in \mathscr{C}$ and $\bar{a} \in M^r$, where $E(\bar{x}_1, \bar{x}_2, \bar{a})$ defines an equivalence relation on M^n. Consider the definable set $\phi(M^n, \bar{b})$, and suppose that $h(\phi(M^n, \bar{b})) = (d, \mu)$. Let $\alpha_\phi := |\phi(M^n, \bar{b})/E_{\bar{a}}|$. Consider first the case where each $E_{\bar{a}}$-class in $\phi(M^n, \bar{b})$ has the same dimension and measure, (e, ν), say. Notice that $e \leq d$. Let $\varepsilon > 0$. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be such that $\left|\frac{\mu \pm \varepsilon_1}{\nu - \varepsilon_2} - \frac{\mu}{\nu}\right| < \varepsilon$ and $\left|\frac{\mu - 2\varepsilon_2 \cdot \frac{\mu}{\nu} \pm \varepsilon_1}{\nu - \varepsilon_2} - \frac{\mu}{\nu}\right| < \varepsilon$.

Then for large enough $M \in \mathscr{C}$, we have

$$(\mu - \varepsilon_1)|M|^{\frac{d}{N}} < |\phi(M^n, \bar{b})| < (\mu + \varepsilon_1)|M|^{\frac{d}{N}}$$

and

$$(\alpha_\phi \cdot \nu - \alpha_\phi \cdot \varepsilon_2)|M|^{\frac{e}{N}} < |\phi(M^n, \bar{b})| < (\alpha_\phi \cdot \nu + \alpha_\phi \cdot \varepsilon_2)|M|^{\frac{e}{N}}$$

So

$$(\alpha_\phi \cdot \nu - \alpha_\phi \cdot \varepsilon_2)|M|^{\frac{e}{N}} < |\phi(M^n, \bar{b})| < (\mu + \varepsilon_1)|M|^{\frac{d}{N}}$$

and thus

$$\alpha_\phi < \frac{(\mu + \varepsilon_1)}{\nu}|M|^{\frac{d-e}{N}} + \alpha_\phi \cdot \frac{\varepsilon_2}{\nu} \tag{5}$$

and similarly

$$\alpha_\phi > \frac{(\mu - \varepsilon_1)}{\nu}|M|^{\frac{d-e}{N}} - \alpha_\phi \cdot \frac{\varepsilon_2}{\nu} \tag{6}$$

Hence (5) gives us that

$$\alpha_\phi < \frac{(\mu + \varepsilon_1)}{\nu}|M|^{\frac{d-e}{N}} + \left(\frac{(\mu + \varepsilon_1)}{\nu}|M|^{\frac{d-e}{N}} + \alpha_\phi \cdot \frac{\varepsilon_2}{\nu}\right) \cdot \frac{\varepsilon_2}{\nu}$$

and similarly for all $r \in \omega$

$$\alpha_\phi < \frac{(\mu + \varepsilon_1)}{\nu}\left(1 + \frac{\varepsilon_2}{\nu} + \cdots + \frac{{\varepsilon_2}^r}{\nu^r}\right)|M|^{\frac{d-e}{N}} + \alpha_\phi \cdot \frac{{\varepsilon_2}^{r+1}}{\nu^{r+1}}$$

and thus taking the limit as $r \to \infty$

$$\alpha_\phi < \left(\frac{\mu + \varepsilon_1}{v - \varepsilon_2}\right) |M|^{\frac{d-e}{N}}$$

This, with (6) gives us that

$$\alpha_\phi > \frac{(\mu - \varepsilon_1)}{v}|M|^{\frac{d-e}{N}} - \left(\frac{\mu + \varepsilon_1}{v - \varepsilon_2}\right)|M|^{\frac{d-e}{N}} \cdot \frac{\varepsilon_2}{v} = \left(\frac{\mu - \varepsilon_1 - 2\varepsilon_2 \cdot \frac{\mu}{v}}{v - \varepsilon_2}\right)|M|^{\frac{d-e}{N}}$$

Thus

$$\left(\frac{\mu - \varepsilon_1 - 2\varepsilon_2 \cdot \frac{\mu}{v}}{v - \varepsilon_2}\right)|M|^{\frac{d-e}{N}} < \alpha_\phi < \left(\frac{\mu + \varepsilon_1}{v - \varepsilon_2}\right)|M|^{\frac{d-e}{N}}$$

and so by choice of ε_1 and ε_2,

$$\left|\alpha_\phi - \frac{\mu}{v}|M|^{\frac{d-e}{N}}\right| < \varepsilon|M|^{\frac{d-e}{N}}$$

Now we turn to the case where the $E_{\bar{a}}$-classes in $\phi(M^n, \bar{b})$ may have different dimensions and measures. However as these classes are all defined by the $\mathscr{L}$-formula $\phi(\bar{x}, \bar{b}) \wedge E(\bar{x}, \bar{u}, \bar{a})$ for varying parameters $\bar{u} \in \phi(M^n, \bar{b})$, there are only finitely many dimension/measure pairs that the classes can take: $(e_1, v_1), \ldots, (e_s, v_s)$ say. Say $M \models \psi_{(e_j, v_j)}(\bar{u}, \bar{a}, \bar{b}) \Leftrightarrow h(\phi(M^n, \bar{b}) \wedge E(M^n, \bar{u}, \bar{a})) = (e_j, v_j)$. Let Y_j be the union of all the $E_{\bar{a}}$-classes in $\phi(M^n, \bar{b})$ of dimension/measure (e_j, v_j), and $\alpha_j := Y_j/E_{\bar{a}}$. Then Y_j is definable by $\phi(\bar{x}, \bar{b}) \wedge \exists \bar{u}(E(\bar{x}, \bar{u}, \bar{a}) \wedge \psi_{e_j, v_j}(\bar{u}, \bar{a}, \bar{b}))$, and $\alpha_1 + \cdots + \alpha_s = \alpha_\phi$. Say $h(Y_j) = (d_j, \mu_j)$.

Let $\varepsilon > 0$. Then by the result obtained above, we have for large enough $M \in \mathscr{C}$,

$$\left|\alpha_j - \frac{\mu_j}{v_j}|M|^{\frac{d_j - e_j}{N}}\right| < \frac{\varepsilon}{s}|M|^{\frac{d_j - e_j}{N}}$$

so

$$\sum_{j=1}^{s} \frac{-\varepsilon}{s}|M|^{\frac{d_j - e_j}{N}} < \alpha_\phi - \sum_{j=1}^{s} \frac{\mu_j}{v_j}|M|^{\frac{d_j - e_j}{N}} < \sum_{j=1}^{s} \frac{\varepsilon}{s}|M|^{\frac{d_j - e_j}{N}}$$

Now set $d' := \mathrm{Max}\{d_j - e_j : 1 \leq j \leq s\}$, $A := \{j : 1 \leq j \leq s \ \ \& \ \ d_j - e_j = d'\}$, and $\mu' := \sum\{\frac{\mu_j}{v_j} : j \in A\}$. Then

$$\sum_{j \in A} \frac{-\varepsilon}{s}|M|^{\frac{d'}{N}} + \sum_{j \notin A}\left(\frac{-\varepsilon}{s} + \frac{\mu_j}{v_j}\right)|M|^{\frac{d_j - e_j}{N}} < \alpha_\phi - \mu'|M|^{\frac{d'}{N}}$$

$$< \sum_{j \in A} \frac{\varepsilon}{s}|M|^{\frac{d'}{N}} + \sum_{j \notin A}\left(\frac{\varepsilon}{s} + \frac{\mu_j}{v_j}\right)|M|^{\frac{d_j - e_j}{N}}$$

and since $|A| \leq s$ and for $j \notin A$ we have $d_j - e_j < d'$ it follows that for large enough $M \in \mathscr{C}$

$$\left|\alpha_\phi - \mu'|M|^{\frac{d'}{N}}\right| < \varepsilon|M|^{\frac{d'}{N}}$$

Furthermore as s is fixed and for each $j \in \{1, \ldots, s\}$ the dimension/measure sets for Y_j are finite, it follows that there are only finitely many possibilities for (d', μ') above.

Now we need to show that the set of $(\bar{b},\bar{a}) \in M^{r+m}$ such that $h(\phi(M^n,\bar{b})/E_{\bar{a}}) = (d',\mu')$ is uniformly definable across $\mathscr{C}$. We know that for any $(d'_j,\mu'_j) \in \{0,\dots,Nn\} \times \mathbf{R}^{>0} \cup \{(0,0)\}$ the set of $(\bar{b},\bar{a}) \in M^{r+m}$ such that $h(Y_j) = (d'_j,\mu'_j)$ is uniformly definable across $\mathscr{C}$. Thus for any assignment of (d'_j,μ'_j) so that $\mu' = \sum\{\frac{\mu_j}{\nu_j} : 1 \leq j \leq s \ \ \& \ \ d'_j - e_j = d'\}$ the set of $(\bar{b},\bar{a}) \in M^{r+m}$ which yield $h(Y_j) = (d'_j,\mu'_j)$ for every j, is also uniformly definable across $\mathscr{C}$. As there are only finitely many such assignments the result follows.

We have shown that, for definable subsets of $M^n/E_{\bar{a}}$, the asymptotic behaviour is definable. Clearly then this holds for definable subsets of $M \cup M^n/E_{\bar{a}}$, and for higher powers we may as usual appeal to proposition 2.2. ■

The following definitions are fairly standard:

3.5. Definition. *Let M and N be structures in first-order languages $\mathscr{L}_{\mathscr{M}}$ and $\mathscr{L}_{\mathscr{N}}$ respectively. We say that M* **is parameter-interpretable (p-interpretable) in** *N if there are $r \in \omega$, an $\mathscr{L}_{\mathscr{N}}$-definable set $X \subseteq N^r$, a $\mathscr{L}_{\mathscr{N}}$-definable equivalence relation E on X, and a map $f : M \to X/E$, and $\mathscr{L}_{\mathscr{N}}$-definable subsets of Cartesian powers of X/E which which interpret the constant, relation, and function symbols of $\mathscr{L}_{\mathscr{M}}$, in such a way that f is an $\mathscr{L}_{\mathscr{M}}$-isomorphism. We write M^* for the $\mathscr{L}_{\mathscr{M}}$-structure induced on X.*

Suppose now that M is p-interpretable in N, via $f : M \to M^*$, and N is p-interpretable in M via $g : N \to N^*$. Then g induces an $\mathscr{L}_{\mathscr{M}}$-isomorphism $g^* : M^* \to M^{**}$ for an $\mathscr{L}_{\mathscr{M}}$-structure M^{**} interpreted in N^* and hence in M. Similarly we get an $L_{\mathscr{N}}$-isomorphism $f^* : N^* \to N^{**}$ where N^{**} is an $\mathscr{L}_{\mathscr{N}}$-structure interpreted in M^* and hence in N. If the isomorphisms $g^*f : M \to M^{**}$, and $f^*g : N \to N^{**}$ are definable in M and N respectively, then we say that M and N are **p-bi-interpretable**. *We say that M is* **$\emptyset$-bi-interpretable** *with N if no parameters from M are involved.*

Notice that being $\emptyset$-bi-interpretable is not symmetric. We rework these in our context:

3.6. Definition. *Let $\mathscr{C}_{\mathscr{M}}$ and $\mathscr{C}_{\mathscr{N}}$ be classes of structures in first-order languages $\mathscr{L}_{\mathscr{M}}$ and $\mathscr{L}_{\mathscr{N}}$ respectively. If there is an injection $i : \mathscr{C}_{\mathscr{M}} \to \mathscr{C}_{\mathscr{N}}$ so that for each $M \in \mathscr{C}_{\mathscr{M}}$, M is p-interpretable in $i(M)$, so that the $\mathscr{L}_{\mathscr{M}}$-structure M^* (i.e X, E, and the $\mathscr{L}_{\mathscr{M}}$-symbols) is uniformly defined across $\mathscr{C}_{\mathscr{M}}$, then we say that $\mathscr{C}_{\mathscr{M}}$* **is p-interpretable in** *$\mathscr{C}_{\mathscr{N}}$.*

Now if $i : \mathscr{C}_{\mathscr{M}} \to \mathscr{C}_{\mathscr{N}}$ is a bijection and for each $M \in \mathscr{C}_{\mathscr{M}}$, M is p-bi-interpretable with $i(M)$, in such away that the structures M^*, N^* and the maps g^*f and f^*g (as above) are uniformly defined across $\mathscr{C}_{\mathscr{M}}$, then we say that $\mathscr{C}_{\mathscr{M}}$ and $\mathscr{C}_{\mathscr{N}}$ are **p-bi-interpretable**. *Again we say that $\mathscr{C}_{\mathscr{M}}$ is* **$\emptyset$-bi-interpretable** *with $\mathscr{C}_{\mathscr{N}}$ if no parameters from $\mathscr{C}_{\mathscr{M}}$ are involved.*

3.7. Lemma. *If $\mathscr{C}_{\mathscr{M}}$ is p-interpretable in $\mathscr{C}_{\mathscr{N}}$, and $\mathscr{C}_{\mathscr{N}}$ is an asymptotic class, then $\mathscr{C}_{\mathscr{M}}$ is a weak asymptotic class. Moreover if $\mathscr{C}_{\mathscr{M}}$ is $\emptyset$-bi-interpretable with $\mathscr{C}_{\mathscr{N}}$, and $\mathscr{C}_{\mathscr{N}}$ is an asymptotic class, then so is $\mathscr{C}_{\mathscr{M}}$.*

Similarly if M and N are infinite structures, M is $\emptyset$-bi-interpretable with N, and N is measurable, then so is M.

Proof. For the first statement, 3.2 and 3.4 give immediately that $\{M^* : M \in \mathscr{C}_{\mathscr{M}}\}$ is a weak asymptotic class, and hence so is $\mathscr{C}_{\mathscr{M}}$.

Now for the second statement we know that $\mathscr{C}_{\mathscr{M}}$ is a weak asymptotic class, and by 3.4 again that $\{i(M) \cup M^* : M \in \mathscr{C}_{\mathscr{M}}\}$ is an asymptotic class. Hence given

$\phi \in \mathcal{L}_M$ and $(d, \mu) \in D$ as in definition 2.1, there is $\phi_{(d,\mu)}(\bar{y}) \in \mathcal{L}_{\mathcal{N}}$ so that $(M, \bar{a}) \in \Phi_{(d,\mu)}$ if and only if $i(M) \models \phi_{(d,\mu)}(f(\bar{a}))$. Of course this holds if and only if $i(M)^* \models \phi_{(d,\mu)}(g^* f(\bar{a}))$, and now as this takes place in M and the $\mathcal{L}_{\mathcal{N}}$-symbols are all interpreted in $\mathcal{L}_{\mathcal{M}}$, we may find a parameter-free $\mathcal{L}_{\mathcal{M}}$-formula $\psi_{(d,\mu)}(\bar{y})$ so that $M \models \psi_{(d,\mu)}(\bar{a})$ if and only if $i(M)^* \models \phi_{(d,\mu)}(g^* f(\bar{a}))$.

The same argument, along with Proposition 5.10 of [5], yields the result for measurable structures. ■

3.8. Corollary. *If $\mathcal{C}_{\mathcal{M}}$ and $\mathcal{C}_{\mathcal{N}}$, in finite languages $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{N}}$ respectively, are p-bi-interpretable, and $\mathcal{C}_{\mathcal{N}}$ is an asymptotic class, then there is an expansion $\mathcal{L}'_{\mathcal{M}}$ of $\mathcal{L}_{\mathcal{M}}$ by finitely many constants, and for each $M \in \mathcal{C}_{\mathcal{M}}$ an expansion M' to $\mathcal{L}'_{\mathcal{M}}$ so that $C'_{\mathcal{M}} := \{M' \colon M \in \mathcal{C}_{\mathcal{M}}\}$ is an asymptotic class.*

Proof. Form $\mathcal{L}'_{\mathcal{M}}$ by adding constants to interpret the parameters needed to define N^* and $g^* f$ uniformly across $\mathcal{C}_{\mathcal{M}}$, and then apply proposition 3.7. ■

§4. Examples.

The main theorem of [1] is exactly the statement that the class of finite fields forms a 1-dimensional asymptotic class.

Similarly in [7] it is shown that any family of finite difference fields

$$\mathcal{C}_{(m,n,p)} := \{(\mathbf{F}_{p^{nk+m}}, \sigma^k) \colon k \in \omega\}$$

where $m, n \in \omega$, $p \in \omega$ is prime, and σ is the Frobenius automorphism, forms an asymptotic class. Notice that in ultraproducts the automporphism interpreted by $\prod_{k\in\omega} \sigma^k/\mathcal{U}$ is a solution of $\sigma^m \cdot \tau^n = \iota$, and hence, in the terminology of Ryten, is a **fractional power of the Frobenius automorphism**.

Now in [8] it has been shown that each family of finite simple groups of fixed Lie-rank is p-bi-interpretable (in the sense of definition 3.6) with either the class of finite fields or one of the classes $\mathcal{C}_{m,n,p}$ as above. Thus by 3.7 and 3.8 each family of finite simple groups is an asymptotic class. For more details see [8].

We now turn to the *Lie-coordinatizable structures*. This is a rich class of $\aleph_0$-categorical supersimple structures which has been thoroughly studied in [2]. One of the main results is that the algebraic characterisation (in terms of coordinatizing Lie-geometries) is equivalent to an abstract notion of being *smoothly approximable*, that is in a strong sense being the infinite limit of a class of finite structures (or *envelopes*). A basic example is that an $\aleph_0$-dimensional vector space is the union of a sequence of finite-dimensional vector spaces (the envelopes) each embedded in the next. For more details see [2].

4.1. Proposition. *Let M be a Lie-coordinatized structure. Then there exists a family $\mathcal{E}$ of finite envelopes for M so that $\mathcal{E}$ smoothly approximates M, and $\mathcal{E}$ is a* rk(M)*-dimensional asymptotic class.*

Proof. Let $\mathcal{E}$ be a maximal family of envelopes for M such that for each dimension corresponding to an orthogonal space, the parity of the dimension is constantly even across $\mathcal{E}$. We shall refine $\mathcal{E}$ in due course.

If $\bar{d} = (d_1, \ldots, d_r)$ is the vector of dimensions assigned by E to each canonical projective geometry respectively, define $\rho'(d_j) := (-\sqrt{q_j})^{d_j}$ where q_j is the size of the base field of the jth canonical projective geometry, or in the disintegrated case $\rho'(d_j) := \sqrt{d_j}$

Then by proposition 5.2.2 of [2], there are $s \in \omega$, $a_1, \dots, a_s \in \boldsymbol{R}$, and $n_{i1}, \dots, n_{ir} \in \omega$ so that for each $E \in \mathscr{E}$,

$$|E| = \sum_{i=1}^{s} a_i (\rho'(d_1))^{n_{i1}} \cdot \dots \cdot (\rho'(d_r))^{n_{ir}}$$

Moreover the proof of 5.2.2 of [2] actually gives that for any definable subset D of M, there are $b_1, \dots, b_s \in \boldsymbol{R}$, and $m_{i1}, \dots, m_{ir} \in \omega$ so that for any $E \in \mathscr{E}$ which contains the parameters for D,

$$|D_E| = \sum_{i=1}^{s} b_i (\rho'(d_1))^{m_{i1}} \cdot \dots \cdot (\rho'(d_r))^{m_{ir}}$$

(It may not be true that the number of terms in the expression for $|D|$ is the same as that for $|E|$, but by setting any extra b_i or a_i to zero, we may assume that both these numbers are equal to s.)

For each $i \leq r$, define $n_i := n_{i1} + \dots + n_{ir}$, and $m_i := m_{i1} + \dots + m_{ir}$. Then, by reordering the a_i and b_i if necessary, we may suppose that that $N := n_1 = n_2 = \dots = n_l > n_i$ for $i > l$, and $e := m_1 = m_2 = \dots = m_k > m_i$ for $i > k$. Notice that proposition 5.2.2 of [2] gives that $a_i > 0$ for $i \leq l$, that $b_i > 0$ for $i \leq k$, that $N = 2\mathrm{rk}(M)$, and that $e = 2\mathrm{rk}(D)$.

With this in mind, if necessary by changing the signs of a_i for $i > l$, and b_i for $i > k$, and by defining $\rho(d_j) := (\sqrt{q_j})^{d_j}$ where q_j is the size of the base field of the jth canonical projective geometry, or in the disintegrated case $\rho(d_j) := \rho'(d_j) = \sqrt{d_j}$, we may rewrite this equation as

$$\frac{|D_E|}{|E|^{\frac{e}{N}}} = \frac{\sum_{i=1}^{s} b_i (\rho(d_1))^{m_{i1}} \cdot \dots \cdot (\rho(d_r))^{m_{ir}}}{\left(\sum_{i=1}^{s} a_i (\rho(d_1))^{n_{i1}} \cdot \dots \cdot (\rho(d_r))^{n_{ir}}\right)^{\frac{e}{N}}}$$

But let us suppose for a moment that we could allow the d_j to range over the positive real numbers. Then for each $x \in \mathbf{R}^{>0}$ we would define $\bar{d}_x := (d_{1x}, \dots, d_{rx})$ where $d_{jx} := \log_{q_j}(x^2)$, unless the jth canonical projective geometry is disintegrated, in which case, $d_{jx} := x^2$. Then we would find that for corresponding notional E_x and $D_x := D_{E_x}$,

$$\begin{aligned}
\frac{|D_x|}{|E_x|^{\frac{e}{N}}} = \alpha_x &:= \frac{\sum_{i=1}^{s} b_i x^{m_{i1}} \cdot \dots \cdot x^{m_{ir}}}{\left(\sum_{i=1}^{s} a_i x^{n_{i1}} \cdot \dots \cdot x^{n_{ir}}\right)^{\frac{e}{N}}} \\
&= \frac{\sum_{i=1}^{k} b_i x^{e} + \sum_{i=k+1}^{r} b_i x^{m_i}}{(\sum_{i=1}^{l} a_i x^{N} + \sum_{i=l+1}^{r} a_i x^{n_i})^{\frac{e}{N}}} \\
&= \frac{\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{r} b_i x^{m_i - e}}{(\sum_{i=1}^{l} a_i + \sum_{i=l+1}^{r} a_i x^{n_i - N})^{\frac{e}{N}}} \\
&\to \frac{(b_1 + \dots + b_k)}{(a_1 + \dots + a_l)^{\frac{e}{N}}}
\end{aligned}$$

as $x \to \infty$ since for $i > k$ we have $m_i - e < 0$ and for $i > l$ we have $n_i - N < 0$.

Now we will approximate this behaviour by suitable choice of natural-number valued $\bar{d}$. For $x \in \boldsymbol{R}$ we write $\alpha(x)$ to mean to the rounding of x to the nearest integer, and $\beta(x) := x - \alpha(x)$. Then for each $Q \in \omega$, we may pick an even natural number $\zeta_Q \in \omega$ so that defining $\varepsilon_j := \beta(\zeta_Q \cdot \log_{q_j}(Q^2))$ for each j corresponding to a canonical projective geometry, $-\frac{1}{Q} < \varepsilon_j < \frac{1}{Q}$. (This is proved by a straightforward induction on the number of such geometries, by considering the image of the map $n \mapsto \beta(n \cdot \log_q(Q^2))$ as a subset of $\left[\frac{1}{2}, \frac{1}{2}\right]$.)

Now define $\bar{d}_Q := (d_{1Q}, \ldots, d_{rQ})$ where $d_{jQ} := \alpha(\zeta_Q \cdot \log_{q_j}(Q^2))$ unless the jth canonical projective geometry is disintegrated, in which case we let $d_{jQ} := Q^{2\zeta_Q}$, and $\varepsilon_j := 0$. Notice that $\bar{d}$ is independent of D. Then we find for each non-disintegrated canonical projective geometry $\rho(d_{jQ}) = q_i^{\frac{1}{2}d_{jQ}} = q_j^{\frac{1}{2}(\zeta_Q \cdot \log_{q_j}(Q^2) - \varepsilon_j)} = Q^{\zeta_Q} \cdot q_j^{-\frac{\varepsilon_j}{2}}$, and similarly for each disintegrated projective geometry $\rho(d_{jQ}) = (d_{jQ})^{\frac{1}{2}} = (Q^{2\zeta_Q})^{\frac{1}{2}} = Q^{\zeta_Q}$. We adopt the convention that if the jth canonical projective geometry is disintegrated then $q_j := 1$.

$$
\begin{aligned}
\frac{|D_Q|}{|E_Q|^{\frac{e}{N}}} &= \frac{\sum_{i=1}^{s} b_i \rho(d_{1Q})^{m_{i1}} \cdot \cdots \cdot \rho(d_{rQ})^{m_{ir}}}{\left(\sum_{i=1}^{s} a_i \rho(d_{1Q})^{n_{i1}} \cdot \cdots \cdot \rho(d_{rQ})^{n_{ir}}\right)^{\frac{e}{N}}} \\
&= \frac{\sum_{i=1}^{s} b_i (q_1^{-\frac{\varepsilon_1}{2}} Q^{\zeta_Q})^{m_{i1}} \cdot \cdots \cdot (q_r^{-\frac{\varepsilon_r}{2}} Q^{\zeta_Q})^{m_{ir}}}{\left(\sum_{i=1}^{s} a_i (q_1^{-\frac{\varepsilon_1}{2}} Q^{\zeta_Q})^{n_{i1}} \cdot \cdots \cdot (q_r^{-\frac{\varepsilon_r}{2}} Q^{\zeta_Q})^{n_{ir}}\right)^{\frac{e}{N}}} \\
&= \frac{\sum_{i=1}^{k} b_i q_1^{-\frac{\varepsilon_1 m_{i1}}{2}} \cdot \cdots \cdot q_r^{-\frac{\varepsilon_r m_{ir}}{2}}}{\left(\sum_{i=1}^{l} a_i q_1^{-\frac{\varepsilon_1 n_{i1}}{2}} \cdot \cdots \cdot q_r^{-\frac{\varepsilon_r n_{ir}}{2}}\right)^{\frac{e}{N}}} + o(Q^{-\frac{1}{N}})
\end{aligned}
$$

Thus as $Q \to \infty$, each $\varepsilon_j \to 0$, so each $q_j^{-\frac{\varepsilon_j m_{ij}}{2}} \to 1$ and $q_j^{-\frac{\varepsilon_j n_{ij}}{2}} \to 1$. Therefore $\frac{|D|}{|E|^{\frac{e}{N}}} \to \frac{(b_1+\cdots+b_k)}{(a_1+\cdots+a_l)^{\frac{e}{N}}}$, as required. So we may say (in the sense of definition 2.1) that $h(D) = (\frac{e}{2}, \frac{(b_1+\cdots+b_k)}{(a_1+\cdots+a_l)^{\frac{e}{N}}})$.

Notice that the dimension/measure of a set is determined by the polynomial which gives its size, and for any envelopes E, E' both containing $\bar{a}$, that $h(\phi(E^n, \bar{a})) = h(\phi(E'^n, \bar{a}))$ as they are given by the same polynomial (again see the beginning of the proof of 2.2 in [2]). For any $E \in \mathscr{E}$, let $\sim$ be the equivalence relation on E^m given by $\bar{y} \sim \bar{y}' :\Leftrightarrow |\phi(E^n, \bar{y})| = |\phi(E^n, \bar{y}')|$. Then in large enough E each $\sim$-class corresponds to a polynomial for $|\phi(E^n, \bar{a})|$. Then in any $E \in \mathscr{E}$, the $\sim$-classes are invariant under $\mathrm{Aut}(E)$, hence they are definable in E over $\emptyset$. But as M is smoothly approximable (see 5 and 7 of definition 2.1.1 of [2]), it follows that the $\sim$-classes are definable in M. Then as M is $\aleph_0$-categorical, there are only finitely many possibilities for $\mathrm{tp}_M(\bar{y})$. This shows that the set of $\sim$-classes, and hence the set of polynomials, and thus the set of dimension-measure pairs for $\phi(\bar{x}, \bar{y})$, is finite.

Finally we must show that dimension/measure is definable. But the $\aleph_0$-categoricity of M yields that the $\sim$-classes are $\emptyset$-definable, given (d_i, μ_i) there is a disjunction $\rho(\bar{y})$ of the $\rho_j(\bar{y})$ so that $h(\phi(E^n, \bar{a})) = (d_i, \mu_i) \Leftrightarrow E \models \rho(\bar{a}))$: (this corresponds to different polynomials which yield the same measure).

Now if $\bar{a} \in \rho(M^m)$ then for all sufficiently large $E \in \mathscr{E}$, we will have that $\bar{a} \in \rho(E^n)$ by smooth approximability. This shows that the dimension and measure of $\phi(E^n, \bar{a})$ is uniformly definable in all sufficiently large E. Then to deal with the finite number of exceptionally small E, for each (d_i, μ_i), we form the disjunction ρ'

of ρ with finitely many formulae of the form $\mathrm{Th}(E) \wedge \mathrm{tp}_E(\bar{a})$ where $\bar{a} \in E^m$ is such that $h(\phi(E^n, \bar{a})) = (d_i, \mu_i)$. ∎

4.2. Corollary. *Lie-coordinatizable structures are measurable.*

Proof. Given a definable set D in a Lie-coordinatized structure, take a collection of envelopes $\mathscr{E}$ as in 4.1, and simply define $h(D)$ as in the proof of 4.1 above. The fact that this function satisfies definition 2.9 is immediate from the fact that $\mathscr{E}$ is an asymptotic class.

Now a structure is Lie-coordinatizable if it is $\emptyset$-bi-interpretable with a Lie-coordinatized structure. By 3.7 the result follows. ∎

In definition 2.1 we deliberately allow the error term of the asymptotic estimates to be as large as possible. In the case of finite fields [1] gives (as in the Lang-Weil estimates) an error term of $C|M|^{d-\frac{1}{2}}$ for some constant $C \in \mathbf{R}^{>0}$. We might therefore define a **Lang-Weil class** to be an N-dimensional asymptotic class where we have error terms at least as tight as $C|M|^{\frac{d}{N}-\frac{1}{2N}}$. In some cases though we can do better than this:

4.3. Proposition. *If a Lie-coordinatized structure M as in the above proposition involves only one canonical projective geometry over a finite field, $\mathbf{F}_q$ say, then, setting $N = 2rk(M)$, we get that for any definable set D there exists $C > 0$ so that for all sufficiently large $E \in \mathscr{E}$,*

$$\left| |D| - meas(D)|E|^{\frac{dim(D)}{N}} \right| < C|E|^{\frac{dim(D)}{N}-\frac{1}{N}}$$

Proof. For $r \in \mathbf{R}$ and $n \in \omega$ we'll use the notation ${}^rC_n$ to denote the generalised binomial coefficient $\frac{r\cdot(r-1)\cdot\dots\cdot(r-n+1)}{n!}$.

Say $|E| = \sum_{i=1}^{s} a_i(\sqrt{q})^{dn_i}$, and $|D| = \sum_{i=1}^{s} b_i(\sqrt{q})^{dm_i}$, where $n_1 > n_2 > \dots > n_s$, and $m_1 > m_2 > \dots > m_s$. Then for constants $c_i \in \mathbf{R}$, and $c', c'', c''', c'''', c''''' > 0$, and large enough d, we get

$$
\begin{aligned}
&\left| |D| - \frac{b_1}{a_1^{\frac{m_1}{n_1}}} |E|^{\frac{m_1}{n_1}} \right| \\
&= \left| \sum_{i=1}^{s} b_i(\sqrt{q})^{dm_i} - b_1(\sqrt{q})^{dm_1}\Big(1 + \sum_{i=2}^{s} c_i(\sqrt{q})^{d(n_i-n_1)}\Big)^{\frac{m_1}{n_1}} \right| \\
&= \left| \sum_{i=2}^{s} b_i(\sqrt{q})^{dm_i} - b_1(\sqrt{q})^{dm_1}\Big(\sum_{k=1}^{\infty} ({}^{\frac{m_1}{n_1}}C_k)\Big(\sum_{i=2}^{s} c_i(\sqrt{q})^{d(n_i-n_1)}\Big)^k\Big) \right| \\
&\qquad \text{and as } m_i \le m_1 - 1 \text{ and } n_i \le n_1 - 1 \text{ for } i \in \{2, \dots, s\} \\
&\le \left| c'(\sqrt{q})^{d(m_1-1)} + b_1(\sqrt{q})^{dm_1}\Big(\sum_{k=1}^{\infty} ({}^{\frac{m_1}{n_1}}C_k)\Big(c''(\sqrt{q})^{-d)}\Big)^k\Big) \right| \\
&\le \left| c'(\sqrt{q})^{d(m_1-1)} + b_1 c''(\sqrt{q})^{d(m_1-1)}\Big(\sum_{k=0}^{\infty} ({}^{\frac{m_1}{n_1}}C_{k+1})\Big(c''(\sqrt{q})^{-d)}\Big)^k\Big) \right| \\
&\le \left| c'(\sqrt{q})^{d(m_1-1)} + b_1 c''(\sqrt{q})^{d(m_1-1)}\Big(\frac{m_1}{n_1}\sum_{k=0}^{\infty}\Big({}^{(\frac{m_1}{n_1}-1)}C_k\Big)\Big(c''(\sqrt{q})^{-d)}\Big)^k\Big) \right| \\
&\le \left| c'(\sqrt{q})^{d(m_1-1)} + c'''(\sqrt{q})^{d(m_1-1)}\Big(1 + c''(\sqrt{q})^{-d}\Big)^{\frac{m_1}{n_1}-1} \right| \\
&\le c''''(\sqrt{q})^{d(m_1-1)} = c''''((\sqrt{q})^{dn_1})^{\frac{m_1-1}{n_1}} \le c'''''|E|^{\frac{m_1-1}{n_1}}
\end{aligned}
$$

■

§5. Dimension versus D-rank. In proposition 2.7 we showed that in asymptotic classes, D-rank is bounded above by dimension. The following demonstrates that this inequality may be strict:

5.1. Example. Let $\mathscr{L} := \langle R \rangle$ where R is a unary predicate. For each $n \in \omega$,

$$
\begin{aligned}
M_n &:= \{0, \dots, n-1\} \times \{0, \dots, n-1\} \\
R(M_n) &:= \{(0, i) \colon 0 \le i \le n-1\}
\end{aligned}
$$

Then if $\mathscr{C} := \{M_n \colon n \in \omega\}$, $\mathscr{C}$ is a 2-dimensional class, but any infinite ultraproduct P is just a set where R picks out an infinite/co-infinite subset, so $D(P) = 1$.

We now turn our attention to those circumstances in which equality does hold.

5.2. Definition. *If $\mathscr{C}$ is an N-dimensional asymptotic class in a language $\mathscr{L}$ which satisfies*:

for every $\phi(x, \bar{y}) \in \mathscr{L}$ and every $r \in \omega$, there are $k, Q \in \omega$, where for each $M \in \mathscr{C}^{\ge Q}$ and each $\bar{a} \in M^m$, there are $\psi(x, \bar{z}) \in \mathscr{L}$, where $l(\bar{z}) = s$ say, and $\{\bar{c}_1, \dots, \bar{c}_r\} \subseteq M^s$ so that if dim$(\phi(M, \bar{a})) = d > 0$, then

- *for each $i \in \{1, \dots, r\}$, $\psi(M, \bar{c}_i) \subseteq \phi(M, \bar{a})$*
- *for each $i \in \{1, \dots, r\}$, dim$(\psi(M, \bar{c}_i)) = d - 1$*
- *$\{\psi(x, \bar{c}_i) \colon 1 \le i \le r\}$ is k-inconsistent*

then we say that $\mathscr{C}$ is **imbricated**.

Moreover, given $\phi(x, \bar{y}) \in \mathscr{L}$, as r ranges over ω, M over $\mathscr{C}^{\geq Q}$, and $\bar{a}$ over M^m, if the set of $\psi(x, \bar{z})$ and k used can be chosen to be finite, then we say that $\mathscr{C}$ is **closely imbricated**.

Notice that imbrication is a condition just on formulae in one variable (plus parameters). The following proposition however shows that (as in 2.2) it has consequences in higher powers, and gives the relationship between (close) imbrication, D-rank, and dimension.

5.3. Proposition. *If $\mathscr{C}$ is such that for any infinite ultraproduct P of members of $\mathscr{C}$, any $\phi(x, \bar{y}) \in \mathscr{L}$, and any $\bar{a} \in P^m$, we have $\dim(\phi(P, \bar{a})) = D(\phi(P, \bar{a}))$, then $\mathscr{C}$ is imbricated.*

Conversely if $\mathscr{C}$ is closely imbricated, then for any infinite ultraproduct P, $\phi(\bar{x}, \bar{y}) \in \mathscr{L}$, and $\bar{a} \in P^n$, we have $\dim(\phi(P^n, \bar{a})) = D(\phi(P^n, \bar{a}))$.

Proof. Suppose first that $\mathscr{C}$ is not imbricated. Then there exist $\phi(x, \bar{y}) \in \mathscr{L}$ and $r \in \omega$ so that for all $k, Q \in \omega$ there are $M_{k,Q} \in \mathscr{C}^{\geq Q}$ and $\bar{a}_{k,Q} \in M^m_{k,Q}$, where for all $\psi(x, \bar{z}) \in \mathscr{L}$ and all $\{\bar{c}_1, \dots, \bar{c}_r\} \subseteq M^s$, at least one of the imbrication properties above fails. Let $\mathscr{U}$ be a non-principal ultrafilter on ω, and let $P := \prod_{Q \in \omega} M_{Q,Q}/\mathscr{U}$, and $\bar{a} := \prod_{Q \in \omega} \bar{a}_{Q,Q}/\mathscr{U}$. Suppose now for a contradiction that $\dim(\phi(P, \bar{a})) = D(\phi(P, \bar{a}))$. Then we can find $k \in \omega$, $\psi(x, \bar{z}) \in \mathscr{L}$, and $\{\bar{c}_i : i \in \omega\} \subseteq P^s$ so that:

- for each $i \in \omega$, $P \models \psi(x, \bar{c}_i) \to \phi(x, \bar{a})$
- for each $i \in \omega$, $D(\psi(P, \bar{c}_i)) = D(\phi(P, \bar{a})) - 1$
- $\{\psi(x, \bar{c}_i) : i \in \omega\}$ is k-inconsistent

Notice that by 2.7 the second of these forces $\dim(\psi(P, \bar{c}_i)) \geq \dim(\phi(P, \bar{a})) - 1$. But then there must exist $U \in \mathscr{U}$ where for all $Q \in U$:

- for $1 \leq i \leq r$, $M_{Q,Q} \models \psi(x, \bar{c}_i(M_{Q,Q})) \to \phi(x, \bar{a}_{Q,Q})$
- for $1 \leq i \leq r$, $\dim(\psi(M_{Q,Q}, \bar{c}_i(M_{Q,Q}))) \geq \dim(\phi(M_{Q,Q}, \bar{a}_{Q,Q})) - 1$
- $\{\psi(x, \bar{c}_i(M_{Q,Q})) : 1 \leq i \leq r\}$ is k-inconsistent

By choosing $Q \in U$ with $Q \geq k$, we get a contradiction by choice of $M_{Q,Q}$.

Conversely suppose that $\mathscr{C}$ is closely imbricated, that $P = \prod_{M \in \mathscr{D}} M/\mathscr{U}$ is a non-principal ultraproduct of members of $\mathscr{C}$, and that $\dim(\phi(P^n, \bar{a})) = d$. We'll show that if $d > 1$, then we can find a uniformly definable k-inconsistent family of $(d-1)$-dimensional subsets of $\phi(P^n, \bar{a})$. We proceed by induction on n. First we assume without loss of generality (by adding constants to the language) that $\bar{a} = \emptyset$.

When $n = 1$ we know that there is $U \in \mathscr{U}$ where for each $M \in U$, $\dim(M) = d$, and by close imbrication that there are $U' \subseteq U$ with $U' \in \mathscr{U}$, $\psi(x, \bar{z}) \in \mathscr{L}$, $I(M) \in \omega$, and $\{\bar{c}_i(M) : i \in I(M)\} \subset M^s$ so as $|M| \to \infty$ in U', $I(M) \to \infty$, and $\{\psi(x, \bar{c}_i(M)) : i \in I(M)\}$ defines an arbitrarily large k-inconsistent family of $(d-1)$-dimensional subsets of $\phi(P)$, just as before. Hence in P, taking $\bar{c}_i := \prod_{i \in I} \bar{c}_i(M)/\mathscr{U}$, $\{\psi(P, \bar{c}_i) : i \in I\}$ is as required.

Now suppose that $n > 1$, and that the result holds for all definable sets of $P, P^2, \dots, P^{n-1}$. As $d > 1$ it must be that $\phi(P^n)$ has an infinite projected image onto some co-ordinate. Without loss of generality, assume that it is the first. That is $\dim(\pi_1(\phi(P^n))) = r > 0$. Define

$$\phi_a(P^{n-1}) := \{(a_2, \dots, a_n) \in P^{n-1} : P \models \phi(a, a_2, \dots, a_n)\}$$

Now as a varies in $\pi_1(\phi(P^n))$, there are finitely many dimension/measure pairs, say $(e_1, \mu_1), \ldots, (e_t, \mu_t)$ which $\phi_a(P^{n-1})$ can take.

For each (e_l, μ_l), define

$$A_l := \{a \in \pi_1(\phi(P^n)) \colon (\text{dim, meas})(\phi_a(P^{n-1})) = (e_l, \mu_l)\}$$

Now if $e_l = d$ for some l, and $a \in A_l$, then by the inductive hypothesis we can find a uniformly definable k-inconsistent family of $(d-1)$-dimensional subsets of $\phi_a(P^{n-1})$, say $\{\psi_a(P^{n-1}, \bar{c}_i) \colon i \in I\}$, and then $\{\{a\} \times \psi_a(P^{n-1}, \bar{c}_i) \colon i \in I\}$ works for $\phi(P^n)$. Hence we may assume that for all l, $e_l < d$.

Now for each $l \in \{1, \ldots, t\}$, define:

$$(f_l, \nu_l) := (\text{dim, meas})(A_l)$$

Then

$$(\text{dim, meas})\Big(\bigsqcup_{a \in A_l} \{a\} \times \phi_a(P^{n-1})\Big) = (e_l + f_l, \mu_l \cdot \nu_l)$$

Moreover $\phi(P^n)$ is the disjoint union over l of all these sets, hence

$$\text{Max}\{e_l + f_l \colon 1 \leq l \leq t\} = d$$

Say this is attained at l'. Then as noted above $e_{l'} < d$, so $f_{l'} \geq 1$. Therefore, by the inductive hypothesis applied to $A_{l'}$ there is a uniformly definable k-inconsistent family $\{\psi(P, \bar{c}_i) \colon i \in I\}$ of $(f_{l'} - 1)$-dimensional subsets of $A_{l'}$. Finally therefore

$$\Big\{ \bigsqcup_{a \in \psi(P, \bar{c}_i)} \{a\} \times \phi_a(P^{n-1})) \colon i \in I \Big\}$$

is a k-inconsistent family of subsets of $\phi(P^n)$, and each one has dimension $e_{l'} + (f_{l'} - 1) = d - 1$, as required.

So we have shown that for $d > 0$, and for any d-dimensional definable set in P, there is $k > 0$ and a uniformly definable infinite k-inconsistent family of $(d-1)$-dimensional subsets. A straightforward induction on d now proves that on all definable sets in P, dimension and D-rank agree. ■

5.4. Corollary. *In any* 1*-dimensional class, dimension and* D*-rank agree.*

Proof. Let $\mathscr{C}$ be a 1-dimensional class. We need only to show that $\mathscr{C}$ is closely imbricated. Let $\phi(x, \bar{y}) \in \mathscr{L}$. Then if $M \in \mathscr{C}$ and $\bar{a} \in M^m$, either $\dim(\phi(M, \bar{a})) = 0$ in which case we have nothing to show, or $\dim(\phi(M, \bar{a})) = 1$. But in this case we can find a family of 0-dimensional subsets of $\phi(M, \bar{a})$ which are pairwise inconsistent, and whose number grows with $|M|$, namely the points of $\phi(M, \bar{a})$, defined by $x = c$. ■

The next is an example of an imbricated class in which dimension and D-rank fail to agree, and shows that the gap between imbrication and close imbrication is genuine.

5.5. Example. Let $\mathscr{L} := \langle R_i \rangle_{i \in \omega}$ where each R_i is a binary predicate. For each $n \in \omega$ define

$$M_n := \{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$$

where

$$M_n \models R_i(a,b) :\Leftrightarrow \begin{cases} a_1 = b_1 = i, \text{ or} \\ a_1 = b_1 \text{ and } a_2 = b_2 \end{cases}$$

where $a = (a_1, a_2)$, $b = (b_1, b_2)$. Then $\mathscr{C}$ is imbricated. But taking a non-principal ultrafilter $\mathscr{U}$ on ω, and defining $P := \prod_{n\in\omega} M_n/\mathscr{U}$ we find that P has D-rank 1, and that $\mathscr{C}$ is not closely imbricated.

§6. Stable classes.

6.1. Proposition. *Let $\mathscr{C}$ be a class of finite $\mathscr{L}$-structures. Then the following are equivalent*:

1. *Every infinite ultraproduct of members of $\mathscr{C}$ is stable.*
2. *For every $\mathscr{L}$-formula $\phi(\bar{x}, \bar{y})$ there is $K \in \omega$ where for each $M \in \mathscr{C}$ there do not exist $\{\bar{a}_i, \bar{b}_i : i \leq K\}$ in M so that $M \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$.*

Proof. Suppose that 2 fails. Then there exists $\phi(\bar{x}, \bar{y})$ so that for each $K \in \omega$ there is $M_K \in \mathscr{C}$ and $\{\bar{a}_{Ki}, \bar{b}_{Ki} : i \leq K\} \subseteq M_K$ where $M_K \models \phi(\bar{a}_{Ki}, \bar{b}_{Kj}) \Leftrightarrow i \leq j$. Let $\mathscr{U}$ be a non-principal ultrafilter on ω, and let $P := \prod_{K\in\omega} M_K/\mathscr{U}$. Define $\bar{a}_i := \prod_{K=1}^{i} \bar{a}_{KK} \times \prod_{K=i+1}^{\infty} \bar{a}_{Ki}/\mathscr{U}$ and $\bar{b}_i := \prod_{K=1}^{i} \bar{b}_{KK} \times \prod_{K=i+1}^{\infty} \bar{b}_{Ki}/\mathscr{U}$. Then $P \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$, so P is unstable.

Conversely suppose 1 fails. Then there exists an unstable infinite ultraproduct P of members of $\mathscr{C}$. Then there are $\phi(\bar{x}, \bar{y})$ and $\{\bar{a}_i, \bar{b}_i : i \in \omega\} \subseteq P$ where $P \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$. Let $K \in \omega$. Then $P \models \bigwedge_{0\leq i\leq j\leq K} \phi(\bar{a}_i, \bar{b}_j) \wedge \bigwedge_{0\leq j<i\leq K} \neg\phi(\bar{a}_i, \bar{b}_j)$, so there exist $U \in \mathscr{U}$ where for all $k \in U$ we have $M_k \models \phi(\bar{a}_i(M), \bar{b}_j(M)) \Leftrightarrow i \leq j$. ■

6.2. Definition. *We call $\mathscr{C}$* **stable** *if it satisfies* 1 *and* 2 *of proposition* 6.1

We now rework the following standard definitions for convenience, see for instance [3]:

6.3. Definition. *A stable structure M is* **functionally unimodular** *if whenever $f_1, f_2 : A \to B$ are definable maps, and k_1, k_2 natural numbers such that for all $b \in B$ we have for each $i \in \{1, 2\}$, $|f_i^{-1}\{b\}| = k_i$, then $k_1 = k_2$.*

A type $p \in S(B)$ in a stable structure M is **multiplicially unimodular** *if whenever $d_1, \dots, d_n, e_1, \dots, e_n \in p(M)$, $\{d_1, \dots, d_n\}$ and $\{e_1, \dots, e_n\}$ are each acl_B-independent, and $acl_B(\bar{d}) = acl_B(\bar{e})$, then $Mult(\bar{d}/\bar{e}) = Mult(\bar{e}/\bar{d})$.*

It is easy to show that measurable structures are functionally unimodular. The following however seems to be absent from the literature:

6.4. Lemma. *If M is a functionally unimodular stable structure and $p \in S(B)$ a minimal type (i.e a stationary type of U-rank 1) in M, then p is multiplicially unimodular.*

Proof. Let p be as in the above definition. We take M to be somewhat saturated, and $B \subseteq M$ to be small.

Let $p'(\bar{x}) := \mathrm{tp}(d_1, \dots, d_n/B) = \mathrm{tp}(e_1, \dots, e_n/B) = p \otimes p \otimes \cdots \otimes p$, and let $q(\bar{x}, \bar{y}) := \mathrm{tp}(d_1, \dots, d_n, e_1, \dots, e_n/B) \Rightarrow p'(\bar{x}) \wedge p'(\bar{y})$. Let $k_1 := \mathrm{Mult}(\bar{e}/\bar{d})$ and $k_2 := \mathrm{Mult}(\bar{d}/\bar{e})$. Let π_1 and π_2 be the projection maps from $q(M^n, M^n)$ onto the first and second n coordinates respectively. Then for $i \in \{1, 2\}$, $Im(\pi_i) = p'(M^n)$ and π_i is everywhere k_i-to-1.

Claim. There is $\phi(\bar{x}, \bar{y}) \in q(\bar{x}, \bar{y})$ so that π_1 and π_2, when extended to $\phi(M^n, M^n)$, have the same range, and are everywhere k_1-to-1 and k_2-to-1 respectively.

The claim finishes the proof, as we may now apply the definition of functional unimodularity to conclude that $k_1 = k_2$.

Proof of Claim. Suppose not. Then for each $\phi(\bar{x}, \bar{y}) \in q(\bar{x}, \bar{y})$ there is $(\bar{a}, \bar{a}_1, \dots, \bar{a}_{k_1+1}, \bar{b}_1, \dots, \bar{b}_{k_2+1})$ from M where

$$M \models \Big(\bigwedge_{1 \le i < j \le k_1+1} \bar{a}_i \neq \bar{a}_j\Big) \wedge \Big(\bigwedge_{1 \le i < j \le k_2+1} \bar{b}_i \neq \bar{b}_j\Big) \tag{7}$$

and

$$\begin{aligned} M \models\ & \big(\exists \bar{y}\phi(\bar{a}, \bar{y}) \wedge \neg\exists \bar{x}\phi(\bar{x}, \bar{a})\big) \vee \big(\neg\exists \bar{y}\phi(\bar{a}, \bar{y}) \wedge \exists \bar{x}\phi(\bar{x}, \bar{a})\big) \\ & \vee \left(\bigvee_{i=1}^{k_1-1} \phi(\bar{a}, M^n) = \{\bar{a}_1, \dots, \bar{a}_i\}\right) \vee \left(\bigvee_{i=1}^{k_2-1} \phi(M^n, \bar{a}) = \{\bar{b}_1, \dots, \bar{b}_i\}\right) \\ & \vee \big(\{\bar{a}_1, \dots, \bar{a}_{k_1+1}\} \subseteq \phi(\bar{a}, M^n)\big) \vee \Big(\{\bar{b}_1, \dots, \bar{b}_{k_2+1}\} \subseteq \phi(M^n, \bar{a})\Big) \end{aligned} \tag{8}$$

So by compactness and saturation, there is $(\bar{a}, \bar{a}_1, \dots, \bar{a}_{k_1+1}, \bar{b}_1, \dots, \bar{b}_{k_2+1})$ from M which satisfies (7) and (8) for every $\phi \in q$.

Case 1: there is $\phi \in q$, where

$$M \models \big(\exists \bar{y}\phi(\bar{a}, \bar{y}) \wedge \neg\exists \bar{x}\phi(\bar{x}, \bar{a})\big) \vee \big(\neg\exists \bar{y}\phi(\bar{a}, \bar{y}) \wedge \exists \bar{x}\phi(\bar{x}, \bar{a})\big).$$

Without loss of generality suppose, that the first disjunct holds. Then for any $\psi \in q$, if $M \models \neg\exists \bar{y}\psi(\bar{a}, \bar{y})$, then $(\phi \wedge \psi)(\bar{a}, M^n) = (\phi \wedge \psi)(M^n, \bar{a}) = \emptyset$ which contradicts the fact that $(\bar{a}, \bar{a}_1, \dots, \bar{a}_{k_1+1}, \bar{b}_1, \dots, \bar{b}_{k_2+1})$ satisfies (8) with $(\phi \wedge \psi)$ in place of ϕ. Hence for all $\psi \in q$, $M \models \exists \bar{y}\psi(\bar{a}, \bar{y})$ and so by compactness and saturation, $\bar{a} \in \pi_1(q(M^n, M^n))$, but as $M \models \neg\exists \bar{x}\phi(\bar{x}, \bar{a})$, we have $\bar{a} \notin \pi_2(q(M^n, M^n))$, which is a contradiction.

Case 2: there is $\phi \in q$, where

$$M \models \left(\bigvee_{i=1}^{k_1-1} \phi(\bar{a}, M^n) = \{\bar{a}_1, \dots, \bar{a}_i\}\right) \vee \left(\bigvee_{i=1}^{k_2-1} \phi(M^n, \bar{a}) = \{\bar{b}_1, \dots, \bar{b}_i\}\right).$$

Again suppose the first disjunct holds. Then for any $\psi \in q$, we have $(\phi \wedge \psi)(\bar{a}, M^n) \subseteq \phi(\bar{a}, M^n)$. If $(\phi \wedge \psi)(\bar{a}, M^n) = \emptyset$ for any $\psi \in q$, then either $(\phi \wedge \psi)(\bar{a}, M^n) = (\phi \wedge \psi)(M^n, \bar{a}) = \emptyset$ which again contradicts (8), or $M \models \exists \bar{y}((\phi \wedge \psi)(\bar{y}, \bar{a}) \wedge \neg(\phi \wedge \psi)(\bar{a}, \bar{y}))$ so case 1 applies with $\phi \wedge \psi$ in place of ϕ. Otherwise $\bar{a} \in \pi_1(q(M^n, M^n))$, but $1 \le |\pi_1^{-1}\{\bar{a}\}| \le k_1 - 1$, which is a contradiction.

Case 3: for all $\phi \in q$, we have

$$M \models \big(\{\bar{a}_1, \dots, \bar{a}_{k_1+1}\} \subseteq \phi(\bar{a}, M^n)\big) \vee \Big(\{\bar{b}_1, \dots, \bar{b}_{k_2+1}\} \subseteq \phi(M^n, \bar{a})\Big).$$

Now suppose that for some $\psi \in q$ we have

$$M \models \neg\Big(\{\bar{b}_1, \dots, \bar{b}_{k_2+1}\} \subseteq \psi(M^n, \bar{a})\Big).$$

Then as $(\phi \wedge \psi)(\bar{a}, M^n) \subseteq \phi(\bar{a}, M^n)$, it must be the case that for all $\phi \in q$ we have $M \models (\{\bar{a}_1, \ldots, \bar{a}_{k_1+1}\} \subseteq (\phi \wedge \psi)(\bar{a}, M^n))$. Therefore $\{\bar{a}_1, \ldots, \bar{a}_{k_1+1}\} \subseteq \pi_1^{-1}\{\bar{a}\}$, which is a contradiction. ■

The final result is in terms of *local modularity*: an important notion in geometric stability theory. A stable structure M is said to be **locally modular** if for any two sets A and B, it holds that A and B are independent over $\operatorname{acl}(A) \cap \operatorname{acl}(B)$. There are several equivalent formulations, see for instance [6] for more details.

6.5. PROPOSITION. *Stable measurable structures are locally modular. In particular, any stable infinite ultraproduct from an asymptotic class is locally modular.*

PROOF. Let M be measurable and stable. Then M is supersimple by 2.7, and of finite U-rank (since $U(M) = D(M)$). As M is stable and supersimple, it is superstable and by 2.5.8 of [6] it suffices to show that any minimal type p in M is locally modular. Moreover M is functionally unimodular, and so by lemma 6.4, p is multiplicially unimodular. But then by 2.4.15 of [6] (see also [3]), p is 2-pseudolinear, and hence by 5.3.2 of [6], locally modular. ■

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